OBSERVATION OPERATORS FOR EVOLUTIONARY INTEGRAL EQUATIONS

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ABSTRACT. We analyze admissibility and exactness of observation operators arising in control theory for Volterra integral equations. We give a necessary and sufficient criterion for an unbounded observation operator to map a solution into L^2. We then discuss the Hautus Lemma, giving a partial result and an example where it fails.

1. INTRODUCTION

We consider control theory for non-autonomous, infinite-dimensional, linear dynamical systems which arise from Volterra integral equations. It is the purpose of this paper to characterize those observation operators which are admissible and exact for such integral equations, thereby extending results from F. Callier and P. Grabowski [1] and K.-J. Engel [2]. Classical theory for C_0-Semigroups considers the system

$$\begin{array}{rcl} x'(t) & = & Ax(t) + Bu(t), \\ y(t) & = & Cx(t), \qquad t \geq 0 \\ x(0) & = & x_0 \end{array} \tag{1.1}$$

where x is a function with values in the state space, u is a function with values in the control space, A describes the internals of the system, B is the control operator, and C is the observation operator.

We consider trivial B ($B = 0$) and non-trivial C, since here we are only interested in the uncontrolled system. We first need the definition of a solution family corresponding to a C_0-Semigroup in the non-scalar Volterra Integral Equation case. Let Y, X be Banach spaces, Y densely embedded in X and let $A \in BV_{\text{loc}}(\mathbb{R}_+, \mathcal{B}(Y, X))$, normalized by $A(0) = 0$ and left-continuity. Looking for solutions of

$$u'(t) = \int_0^t dA(t-s)u(s), \qquad t \geq 0; u(0) = u_0 \tag{1.2}$$

we define the solution family.

Definition 1.1. $S : \mathbb{R}_+ \to \mathcal{B}(X)$ is called the *solution family* to (1.2), if

(i) $\mathbb{R}_+ \ni t \mapsto S(\cdot)x \in X$ is continuous for $x \in X$ and $S(0) = I$.

(ii) $S(t)Y \subset Y$ for $t \geq 0$ and $\mathbb{R}_+ \ni t \mapsto S(\cdot)y \in Y$ is continuous for $y \in Y$.

(iii) for all $T > 0$ holds $S(\cdot)y \in W^{1,\infty}([0, T], X)$, $S'(t)y = (dA * S)(t)y$, and $S'(t)y = (S * dA)(t)y$ for $y \in Y$ and $t \in (0, T)$.

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We now consider the control problem in the Banach spaces Y and X with normalized $A \in BV_{\text{loc}}(\mathbb{R}_+, \mathcal{B}(Y, X))$:

$$\begin{aligned} u'(t) &= (dA * u)(t), \\ y(t) &= Cu(t), \qquad t \geq 0, \\ u(0) &= u_0. \end{aligned} \tag{1.3}$$

The operator $C : Y \to Z$ is the observation operator, where Z is a third Banach space, called the control space.

Let $A \in BV_{\text{loc}}(\mathbb{R}_+, \mathcal{B}(Y, X))$, such that

$$\int_0^\infty e^{-\epsilon t} \|dA(t)\| < \infty$$

for all $\epsilon > 0$. A is then called *exponentially stable*. We then define the Laplace transform $\mathcal{L}$ by

$$\mathcal{L}(dA)(z) = \int_0^\infty e^{-zt} dA(t), \qquad z \in \mathbb{C}_+. \tag{1.4}$$

Let $e_0(t) = 1$ for $t > 0$, 0 for $t \leq 0$. e_0 is called the Heaviside function.

2. Admissibility

We are interested in showing a characterization of observability for Volterra integral equations, similar to the one proven for C_0-Semigroups in [1] (for a simplified version see [2]). We do not have the same methods at hand, but some theorems remain valid. Notationally we let $L^2(\mathbb{R}_+, X) = L^2(X)$ for any Banach space X and handle Sobolev spaces similarly.

Definition 2.1. The operator C is called an *admissible* observation operator, if there exists a $\Gamma > 0$, such that

$$\int_0^\infty \|CS(t)y\|_Z^2 dt \leq \Gamma \|y\|_X^2$$

for all $y \in Y$.

The latter implies that $X \supset Y \ni y \mapsto CS(\cdot)y \in L^2(Z)$ is bounded. We define $Py = CS(\cdot)y$ and remark that $P : Y \subset X \to L^2(Z)$ is bounded if and only if C is admissible.

We will consider an extended dynamical system. Define $(R_t f)(x) = f(t-x)$ the reflection operator, and $T_t f(x) = f(x-t)$ the shift operator for any $f : \mathbb{R}_+ \to X$ (these image functions are trivially extended to $\mathbb{R}_+$!) and $x, t \geq 0$.

$$\begin{aligned} X_e &= X \times L^2(Z), \\ Y_e &= \left\{(y, f) \in X_e : y \in Y; f \in W^{1,2}(Z); f(0) = Cy\right\}, \\ A_e(t)(y, f) &= (A(t)y, -e_0(t)f') \quad \text{for } (y, f) \in Y_e, \qquad t \geq 0, \\ \widetilde{S}_e(t)(y, f) &= (S(t)y, R_t Py + T_t f) \quad \text{for } (y, f) \in Y_e, \qquad t \geq 0. \end{aligned}$$

Consider the thus extended problem in X_e:

$$\begin{aligned} u_e'(t) &= (dA_e * u_e)(t), \qquad t \geq 0, \\ u_e(0) &= u_{e,0}. \end{aligned} \tag{2.1}$$

Using the fact that $W_0^{1,2}(Z)$ is dense in $L^2(Z)$ it is easy to see that Y_e is dense in X_e. We are now in a position to formulate the central proposition.

Proposition 2.2. Assume that S is a bounded solution family to (1.2). Then (2.1) admits a bounded solution family S_e if and only if C is admissible.

Proof. We start with the if part. Since C is admissible, P is bounded and $\widetilde{S}_e$ has a continuous extension in X_e we call S_e. We now prove that S_e is indeed the bounded solution family to (2.1) according to Definition 1.1.

It is obvious that S_e is strongly continuous on X_e and that $S_e(0) = I$. Therefore (i) holds. Now, S is strongly continuous on Y and $(R_tPy + T_tf)(0) = CS(t)y$. To show (ii), $S_e(t)Y_e \subset Y_e$ and the strong continuity in Y_e, it remains to show that $g : \mathbb{R}_+ \to W^{1,2}(Z)$, with $g(t) := R_tPy + T_tf$ for $(y, f) \in Y_e, t \geq 0$ is continuous. Let $t, h > 0$.

$$\begin{aligned} & h^{-1}(R_tPy + T_tf)(s + h) - (R_tPy + T_tf)(s) \\ = \quad & h^{-1} \begin{cases} CS(t - s - h)y - CS(t - s)y & t \geq s + h, \\ f(s + h - t) - CS(t - s)y & s + h > t \geq s, \\ f(s + h - t) - f(s - t) & s > t. \end{cases} \end{aligned}$$

Since $f(0) = Cy$ this converges in the $L^2(Z)$-norm as $h \to 0$. But from this not only $g(t) \in W^{1,2}(Z)$ follows, but $\mathbb{R}_+ \ni t \mapsto g(t)' \in L^2(Z)$ is continuous, in fact

$$g(t)' = \begin{cases} -(P * dA)(t - s)y & t > s, \\ f'(s - t) & t < s. \end{cases} \quad \text{for almost all } s > 0.$$

Therefore g is to continuous. We now prove (iii) in a similar fashion. We have

$$\begin{aligned} & h^{-1}((R_{t+h}Py + T_{t+h}f)(s) - (R_tPy + T_tf)(s)) \\ = \quad & h^{-1} \begin{cases} CS(t + h - s)y - CS(t - s)y & t > s, \\ CS(t - s)y - f(s - t) & t + h > s \geq t, \\ f(s - t - h) - f(s - t) & s \geq t + h, \end{cases} \end{aligned}$$

which again shows convergence of $h^{-1}(R_{t+h}Py + T_{t+h}f - R_tPy + T_tf)$ in $L^2(Z)$ as $h \to 0$ to

$$\frac{dg(t)}{dt}(s) = \begin{cases} (P * dA)(t - s)y & t > s, \\ -f'(s - t) & t < s. \end{cases}$$

Moreover, combining this with the structure of S_e and the fact that S is a solution family, we obtain the convergence of $h^{-1}(S_e(t + h)) - S_e(t))u_{e,0}$ to

$$((S * dA)(t)y, (P * dA)(t)y + T_tf') = (S_e * dA_e)(t)u_{e,0}$$

as $h \to 0$. ($u_{e,0} = (y, f) \in Y_e$.) Immediately the two equations from (iii) follow for A_e and S_e. Finally, S_e is bounded:

$$\|S_e(t)u_{e,0}\| \leq \|S(t)\| \, \|y\| + \Gamma\|y\| + \|f\| \leq (\|S(t)\| + \Gamma + 1)\|u_{e,0}\|,$$

since S is a bounded solution family to (1.2).

We turn to the only if part of the proposition. First note that

$$S_e(t)(y, f) = (S(t)y, F_ty + T_tf), \qquad (y, f) \in X \times L^2(Z),$$

where $F_t : X \to L^2(Z)$ for $t > 0$ is bounded, say by $M > 0$. On the other hand, $\widetilde{S}_e(t) : Y_e \to X_e$, can be differentiated in X_e for each $u_{e,0} \in Y_e$ and satisfies $\widetilde{S}_e'(t)u_{e,0} = (dA_e * \widetilde{S}_s)(t)u_{e,0}$. By uniqueness it follows that $\widetilde{S}_e(t) \subset S_e(t)$ and consequently

$$\int_0^t \|CS(t)y\|^2 ds = \|R_tPy\|_2^2 = \|F_tu\|_2^2 \leq M\|y\|.$$

□

The structure of S_e can be exploited to yield the following theorem.

Theorem 2.3. *Assume that A is exponentially stable the solution family S to (1.2) is bounded. Then C is admissible if and only if there exists an $M \geq 0$ such that for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}_+$ holds*

$$\| \left(\frac{d}{d\lambda}\right)^n e^{-\lambda \cdot} C(\lambda - \mathcal{L}(dA)(\lambda))^{-1} y\| \leq \frac{Mn!}{(\Re(\lambda))^n} \|y\|$$

for $y \in X$ in $L^2(Z)$ and for $y \in Y$ in $W^{1,2}(Z)$ (with norms in the corresponding spaces).

Proof. By the Generation Theorem 6.3 [4] of J. Prüss, S_e is a bounded solution family, if and only if $\|(d/d\lambda)^n \lambda^{-1}(I - \mathcal{L}(A_e)(\lambda))^{-1}\| \leq M_0 n!(\Re(\lambda))^{-n}$ for all $\lambda \in \mathbb{C}_+$ in $\mathcal{B}(X_e)$ and $\mathcal{B}(Y_e)$. (We are a bit imprecise here: we need to take the closure of the inverse in $\mathcal{B}(X_e)$). Calculating the norm of the right hand side and using the structure of A_e, we obtain its boundedness if and only if

$$\| \left(\frac{d}{d\lambda}\right)^n \begin{pmatrix} (\lambda - \mathcal{L}(dA)(\lambda))^{-1} & 0 \\ Ce^{-\lambda \cdot}(\lambda - \mathcal{L}(dA)(\lambda))^{-1} & e^{-\lambda \cdot} * \end{pmatrix} \begin{pmatrix} y \\ f \end{pmatrix} \|$$
$$\leq \frac{Mn!}{(\Re(\lambda))^n} \| \left(y, f\right) \|.$$

in X_e and Y_e. To obtain the claim, observe that the only term in the matrix whose boundedness needs to be required is found in the lower left corner of the matrix. □

Unfortunately we need estimates in two spaces, $L^2(Z)$ and $W^{1,2}(Z)$, in contrast to the C_0-Semigroup case ([1], Theorem 2.3). See however [4] for situations, where the second one may be omitted. A convenient necessary condition is given in

Corollary 2.4. Let S be a bounded solution family to (1.2). If C is admissible, then there exists an $M > 0$. such that $\|C\lambda(I - \mathcal{L}(dA)(\lambda))^{-1}\| \leq M(\Re(\lambda))^{-1/2}$ in $\mathcal{B}(X, Z)$.

Proof. If we choose $n = 0$ in the Theorem 2.3, we obtain the stated formula. □

3. Exactness

We first give the definition of exactness, guaranteeing invertibility of the closure of $P : X \supset Y \ni y \to CS(\cdot)y \in L^2(Z)$. While the admissibility condition ensures that all states are observable (and that P is closable), exactness ensures that all states are distinctly observable. Recall Definition 1.1 for solution families.

Definition 3.1. The admissible control operator C is called an *exact* observation operator, if there exists a $\gamma > 0$, such that

$$\int_0^\infty \|CS(t)y\|_X^2 dt \geq \gamma \|y\|_X^2$$

for all $y \in Y$.

A general condition with respect to exactness for C_0-Semigroups is given in [1]. Here we show that a "simple" criterion cannot hold for Volterra integral equations. The well-known Hautus lemma states, that in a finite dimensional setting an operator C is exactly observable, if and only if for all $x \in X$ and $\lambda \in \mathbb{C}$ holds

$\|Ax - \lambda x\| + \|Cx\| > 0$. We first prove that this is necessary, if $A(t) = a(t)A$ (even in the infinite-dimensional case) and then give a counter-example that it is not sufficient. Note that a sufficient condition seems to be unknown (cf. D. Russell and G. Weiss [5]) in the infinite-dimensional C_0-Semigroup case, although a modified condition is there conjectured to be equivalent to exactness.

The solution family S is called exponentially stable, if there exist $M, \omega > 0$, such that $\|S(t)\| \leq Me^{-\omega t}$ for all $t \geq 0$. Note the subtle difference between exponential stability for functions in $BV_{\mathrm{loc}}(\mathbb{R}_+, X)$ and solution families.

Proposition 3.2. Let $A(t) = a(t)A$, where $a \in BV_{\mathrm{loc}}(\mathbb{R}_+)$ is exponentially stable and the solution family S is exponentially stable. If C is admissible and exact, then

$$\|Cx\| + \|(\lambda - A)x\| > 0 \tag{3.1}$$

for all $x \in D(A)$ and $\lambda \in \mathbb{C}$.

Proof. Assume to the contrary that $Cx = 0$ and $Ax = \lambda x$ for some $\lambda \in \mathbb{C}$ and $x \in D(A)$. Then $S(t)x = s_\lambda(t)x$, where s_λ solves the scalar equations ${s_\lambda}' = \lambda da * s_\lambda$ and $s_\lambda(0) = 1$ (cf. [4]). But then $CS(t)x = s_\lambda(t)Cx = 0$ and C is not exact. □

Note that the modified version of the Hautus Lemma of [5] is not easily adapted from the C_0-Semigroup case, since the L^1- and L^2-norms of the scalar functions $s_\mu(t)$ need not have the same growth bound with respect to μ as those of $e^{\mu t}$. There is not much hope that the same conditions are true for the time-independent and the time-dependent situation. For a more detailed comparison in the case of control operators see [3].

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