Perturbation Results for Evolution Equations

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Abstract

We study certain conditions of compatibility between evolution families and spaces, which yield perturbation results for evolution families in a general setting. A theorem for stability of these conditions is given. We compare our results for evolution semigroups and evolution families. Additionally, we apply the results to parabolic problems.

1 Introduction

In recent years W. Desch, W. Schappacher, and others have established multiplicative perturbation results for semigroup theory (e. g. [3], [4], [10]). The method used relies on the variation of constants formula, which can be applied in some cases, where the perturbing operator is not bounded. In particular, they involve the so-called (Z)-condition to obtain generation results for the operators

$$A(I+B), \qquad (I+B)A,$$

where A generates a semigroup. A similar method was employed by the author using a "dual" condition, the (Z^*)-condition, to obtain further results in that direction. In the mean time the results have been extended to cosine families by A. Piskarev and S.-Y. Shaw ([18], [19]) and to certain non-autonomous evolution equations by W. Desch, W. Schappacher, and K. P. Zhang [5], i. e. the question, whether

$$A(I+B(t)), \qquad (I+B(t))A$$

(weakly) generates evolution families was addressed. Even for a non-linear family $B(\cdot)$ generation results were obtained ([11]). The non-linear case will not be dealt with in this paper, however. Applications of a wide variety were considered, ranging from lower order differential terms as perturbation to population

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dynamics and (non-linear) delay differential equations. We will extend these applications to potentials for the Heat equation.

The relationship between (Z)- and (Z^*)-condition is not as simple as a casual view on the definitions would suggest, they present two distinct ways to perturbations. Also they have not been extended to the case, where A is not simply a generator but a family of operators. Since it is our aim to apply and extend the results given to

$$A(t)(I + B(t)), \qquad (I + B(t))A(t),$$

we need to do so. The third section will deal with the main theorems for the non-autonomous Cauchy problem, thereafter we shall examine how the adjusted conditions behave under perturbations. The last two sections contain comparisons with another setting used by Y. Latushkin, R. Nagel and others as well as some applications for certain evolution families.

By the formulas $A(I + B) = A + AB$ and $(I + B)A = A + BA$, understood in a formal sense, the multiplicative perturbations may be understood as additive perturbations. All the results below can be formulated in that sense; the examples in the last section show how the formal sense can be made rigorous. But the conditions are easier to formulate and sometimes it is convenient to see how the domain or the range changes, which is why we choose to write the perturbation the multiplicative way.

2 Preliminaries

Let $A(t), t \in [0, T]$ be a family of linear operators in the Banach space X. An evolution family represents the unique solution to a Cauchy problem such as $u'(t) = A(t)u(t)$ for $t \in [0, T]$. It can be represented as family of continuous linear operators $U(t, s)x$ for an initial value $u(s) = x$, which we call an evolution family. We will only consider evolution families, for which $(t, s) \mapsto U(t, s)x$ is continuous for all $x \in X$ (i. e. strongly continuous). Evolution families satisfy $U(t, s)U(s, r) = U(t, r)$ for $0 \leq s \leq r \leq t \leq T$ (called the Kolmogorov equation) and $U(t, t) = I$ for $t \in [0, T]$. For a general introduction to this theory see A. Pazy [17]. $U(t, s)$ is said to be *(strongly) generated*, if

$$\partial_1 U(t, s)x = A(t)U(t, s)x \tag{1}$$

is continuous in $t \in [0, T]$ for all $x \in D(A(s))$ and $0 \leq s \leq t$. This implies $U(t, \cdot)$ is right differentiable and

$$\partial_2^+ U(t, s)x = -U(t, s)A(s)x.$$

This follows from the Kolmogorov equation and continuity of $\partial_1 U(t, s)x$. Note that $\partial_2^+ U(t, s)x$ is continuous in the first argument for all $x \in D(A(s))$ and $0 < s < t \leq T$. For each $A(t)$ let $A(t)^*$ be some adjoint mapping. The evolution family $U(t, s)$ is said to be *weakly generated*, if there exists a weak*-dense set

$D \subset \cap_{t\in[0,T]} D(A(t)^*)$, such that

$$\partial_2 < U(t,s)x, x^* >= - < x, A(s)^* U(t,s)^* x^* >$$

exists and is continuous in $(s,t) \in \{(\sigma,\tau) : 0 \le \sigma < \tau \le T\}$ for $x \in X$, $x^* \in D$ (with $0 < s < t \le T$). In that case it follows from the Kolmogorov equation that

$$\partial_1^+ < U(t,s)x, x^* >=< U(t,s)x, A(t)^* x^* >$$

exists and is continuous in $t \in [0,T]$ for all $x \in X$, $x^* \in D$, and $0 \le s \le t$. Another notion is quasi-generation: $U(t,s)$ is called *quasi-generated* by $A(t)$, if (1) holds for an extension of $A(t)$. The statements on $\partial_2 U(t,s)x$ still remain true with unextended $A(s)$.

For some types of evolution families these concepts coincide. C_0-semigroups are weakly and strongly generated. If we have two evolution families, U_1 and U_2, generated by the same operator family A, then both are equal. To see this, we differentiate $f(r) := U_1(t,r)U_2(r,s)x$ for $x \in D(A(s))$ with respect to $r \in (s,t)$. But $f'(r) = 0$, hence $f(s) = U_1(t,s)x = U_2(t,s)x = f(t)$ by continuity.

In general strong generation need not imply weak generation, since no appropriate weak*-dense set D has to exist. Also weak generation alone does not guarantee uniqueness either way. The operators $A_0 f = A_1 f = \partial_x f$ in $C([0,1])$ with $D(A_0) = C^1([0,1])$ and $D(A_1) = \{f \in C^1([0,1]) : f(0) = 0\}$ both weakly generate the evolution families

$$(U_0(t,s)f)(x) = \begin{cases} f(x+t-s) & x+t-s \in [0,1] \\ f(1) & \text{otherwise} \end{cases}$$

and

$$(U_1(t,s)f)(x) = \begin{cases} f(x+t-s) & x+t-s \in [0,1] \\ f(1)e^{x+t-s-1} & \text{otherwise} \end{cases},$$

$0 \le s \le t \le T$, $x \in [0,1]$, and $f \in C([0,1])$. We may choose $D = C_C^\infty((0,1))$ (the set of functions with compact support, infinitely differentiable), which, interpreted as a set of density functions, is weak*-dense in $M_{reg}([0,1]) = C([0,1])^*$. The set D renders boundary conditions obsolete.

We make some general Assumptions and notational conventions, which are to hold throughout the remaining article. Suppose we have an operator family $A = \{A(t) : 0 \le t \le T\}$ that generates an evolution family $U = \{U(t,s) : 0 \le s \le t \le T\}$. We set $\Delta_{ST} = \{(s,t) : S \le s \le t \le T\}$ and $\Delta_T = \Delta_{0T}$. For some compact Hausdorff set K and Banach spaces X, Y, the space $C(K, \mathcal{B}_s(X,Y))$ denotes the (Banach) space of strongly continuous functions $f : K \to \mathcal{B}_s(X,Y)$ supplied with the supremum norm $\|\cdot\|_\infty$. Mutatis mutandis we have $L_p(K, \mathcal{B}(X,Y))$, if K is a measure space and $1 \le p \le \infty$. Integrals are always understood to be Bochner integrals. A subset $D \subset X^*$ is *norming*, if $|x| = \sup_{x^*\in D} |< x, x^* >|$ for all $x \in X$.

Definition 2.1 *Let $U = \{U(t,s) : (s,t) \in \Delta_T\}$ be an evolution family in X. Let $A = \{A(t) : t \in [0,T]\}$ be a family of (possibly non-continuous) linear operators in X. Suppose Z is a Banach space embedded in X, such that* *(i) either (a) for all $(s,t) \in \Delta_T$ there is a dense set $D^Z_{s,t}$ in $C([s,t], \mathcal{B}_s(X,Z))$ and for $\Phi \in D^Z_{s,t}$ and $x \in X$, the function $A(\cdot)\Phi(\cdot)x$ is integrable; or (b) there exists a norming set $D^* \subset X^*$, such that for all $(s,t) \in \Delta_T$ there exists a dense set $D^Z_{s,t}$ in $C([s,t], \mathcal{B}_s(X,Z))$ and for $\Phi \in D^Z_{s,t}$, $x^* \in D^*$, and $x \in X$, the function $< \Phi(\cdot)x, A(\cdot)^* U(r,\cdot)^* x^* >$ is integrable for all $r \in [s,t]$,*

(ii) the linear mapping $F_{A,s,t} : D^Z_{s,t} \to C([s,t], \mathcal{B}_s(X))$ defined by

$$< F_{A,s,t}(\Phi)(r)x, x^* >= \int_s^r < \Phi(r')x, A(r')^* U(r,r')^* x^* > \, dr',$$

for $\Phi \in D^Z_{s,t}$, $x^* \in D^*$ (= X^* in case (i) (a)), and $x \in X$ can be extended to $C([s,t], \mathcal{B}_s(X,Z))$,

(iii) $D^Z_{s,t}$ is dense for all $(s,t) \in \Delta_T$ and there exists a continuous bijection $\gamma : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ with $\gamma(0) = 0$ and $\|F_{A,s,t}\| \leq \gamma(t-s)$ for all $(s,t) \in \Delta_T$. In particular, $F_{A,s,t}$ is continuous.

Then we call (A,U,Z) right-compatible.

To distinguish between (i) (a) and (i) (b), we call the former strong and the latter weak compatibility. We do not assume the domains $D(A(t))$, $t \in [0,T]$ to have a common dense subspace, it may well be that $D(A(t)) \cap D(A(s)) = \{0\}$ for $s \neq t$. We also like to consider spaces in which the domains are not dense, this is why we have included a weak version of right-compatibility. The operator family A will always be some sort of generator and in many cases determines U uniquely.

Remark 2.2 We let $D^Z := D^Z_{0,T}$, which is a subset of $D^Z_{s,t}$ for all $(s,t) \in \Delta_T$. We may also extend the elements of D^Z to turn D^Z into a set contained in $C_0(\mathbb{R}, Y)$ (the space of functions vanishing at infinity).

Definition 2.1 more or less coincides with the (Z)-condition given previously ([4], for the operator-valued version cf. [10]) as the following informal discussion illustrates. With the (Z)-condition one requires that $\int_0^t T(s)B\Phi(s)ds$ lies in $D(A_0)$, where A_0 is the (closed) generator of T and then estimates $A_0 \int_0^t T(s)B\Phi(s)ds$. The non-autonomous case requires one to take $A(\cdot)$ under the integral a priori, if we aim for the variation of constants formula. This is in contrast to the semigroup case, where $T(s)A \subset AT(s)$ can be used for all $s \geq 0$. That such an argument is not possible in the non-autonomous case will become clear through the following lemma.

Lemma 2.3 *Suppose U is an evolution family generated (weakly or strongly) by A on the set Δ_T and that $D \subset X$ is dense. If $A(t)U(t,s)x = U(t,s)A(s)x$ for $x \in D$, $(s,t) \in \Delta_T$, then U is a semigroup and $A(t) = A(0)$ for $t \in [0,T]$, where $A(0)$ is a restriction of the generator of the semigroup U.*

Proof: Let $f_{x,x^*}(t,s) =< U(t,s)x, x^* >$ for $x \in D$, $x^* \in D^*$, and $(s,t) \in \Delta_T$. D^* is the weak* dense set in X^*, which is given in the case of a weakly generated

U and can be chosen as X^* in the case U is strongly generated. We have $\partial_1^+ f(t,s) = -\partial_2 f(t,s)$. Since this function is continuous, $\partial_1 f(t,s) = \partial_1^+ f(t,s)$ exists. By an elementary method of characteristics, we have $f(t,s) = g(t-s)$ for some g, In other words,

$$< U(t,s)x, x^* >=< U(t-s,0)x, x^* > .$$

But x, x^* were chosen arbitrary and D is dense in X, D^* is weak*-dense in X^*, thus $U(\cdot,0)$ is a semigroup by the Kolmogorov equation. That A is constant and a subset of the generator of $U(\cdot,0)$ is a trivial consequence of this. □

The classical examples of spaces Z, which make (A,U,Z) right-compatible in the autonomous case are the the Favard class of A and, if A is generator of a holomorphic semigroup, most interpolation spaces between X and $D(A)$. See the last section for details, where these are extended to the non-autonomous case.

Remark 2.4 Sometimes it is convenient to have a pointwise version of Definition 2.1. Under the given hypothesis we may require (i) that there exists a dense set $\widetilde{D}^Z$ in $C([s,t],Z)$, such that for $\phi \in \widetilde{D}^Z$, the function $A(\cdot)\phi(\cdot)$ is integrable and the linear mapping $\widetilde{F}_{A,s,t} : \widetilde{D}^Z \to C([s,t],X)$ defined by

$$\widetilde{F}_{A,s,t}(\phi)(r) = \int_s^r U(r,r')A(r')\phi(r')\,dr',$$

is extendible to $C([s,t],Z)$, with $\|\widetilde{F}_{A,s,t}\| \le \gamma(t-s)$.

By choosing $\phi := \Phi(\cdot)x$, it is obvious that the conditions on $\widetilde{F}_{A,s,t}$ imply those on $F_{A,s,t}$. The other situation can be seen by setting $\Phi_{x^*}(\cdot)x =< x, x^* > \phi$ for $x^* \in X^*$. The short proof does not differ from the one in the semigroup case given by the author (cf. [9], [10]).

Remark 2.5 In case U is a semigroup and $A(0)$ its generator, we may assume that $D(A(0)) \subset i(Z)$, where $i : Z \to X$ is the embedding mapping. $X_{A(0)} = D(A(0))$ with its graph norm makes $(A,T,X_{A(0)})$ right-compatible (cf. [3]). Thus $(A,T,D(A(0)) \oplus i(Z))$ is right-compatible if (A,T,Z) is.

We continue to omit the embedding mapping i, as in most applications Z will be a subspace of X suitably normed. We now turn to the appropriate extension of the (Z^*)-condition.

Definition 2.6 *Let $U = \{U(t,s) : (s,t) \in \Delta_T\}$ be an evolution family in X. Let $A = \{A(t) : t \in [0,T]\}$ be a family of (possibly non-continuous) linear operators in X. Suppose X is embedded into a Banach space Z, such that*

(i) there exists a dense set $D \subset X$, such that for all $(s,t) \in \Delta_T$ there exists a set $D^Z_{s,t}$ in $C([s,t],\mathcal{B}_s(Z,X))$, and for $\Phi \in D^Z_{s,t}$ and $x \in D$ the function $\Phi(\cdot)A(\cdot)U(\cdot,s)x$ is integrable.

(ii) the linear mapping $G_{A,s,t}: D^Z_{s,t} \to C([s,t], \mathcal{B}_s(X))$ *defined by*

$$G_{A,s,t}(\Phi)(r)x = \int_s^r \Phi(r')A(r')U(r',s)x\,dr',$$

for $\Phi \in D^Z_{s,t}$ *and* $x \in D$ *can be extended to* X *and* $C([s,t], \mathcal{B}_s(Z,X))$*,*

(iii) $D^Z_{s,t}$ *is dense and there exists a continuous bijection* $\gamma : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ *with* $\gamma(0) = 0$ *and* $\|G_{A,s,t}\| \le \gamma(t-s)$*. In particular,* $G_{A,s,t}$ *is continuous.*

Then we call (A,U,Z) left-compatible*.*

As in the case of right-compatibility in Remark 2.2, we drop the indices s and t from $D^Z_{s,t}$. Definition 2.6 replaces the (Z^*)-condition in the same manner as the previously given replaces the (Z)-condition.

In both cases, the denseness of the sets D^Z ensure that the mappings $F_{A,s,t}$, resp. $G_{A,s,t}$ overlap properly for different values of $t \in [0,T]$, i. e.

$$F_{A,s,t}(\Phi)(r) = F_{A,s,t'}(\Phi)(r) \tag{2}$$

for $0 \le s \le r \le t \le t' \le T$. Suppose we have a family $B \subset C([0,T], \mathcal{B}_s(X,Z))$. Then it is possible to construct a linear, continuous (well-defined) mapping $F_{A,B} : C(\Delta_T, \mathcal{B}_s(X)) \to C(\Delta_T, \mathcal{B}_s(X))$ by

$$F_{A,B}(\Phi)(t,s) = F_A \circ B(\Phi)(t,s) := F_{A,s,t}(B(\cdot)\Phi(\cdot,s))(t).$$

Accordingly for left-compatibility, if $B \subset C([0,T], \mathcal{B}_s(Z,X))$, we have

$$G_{A,B}(\Phi)(t,s) = G_A \circ B(\Phi)(t,s) := G_{A,s,t}(\Phi(t,\cdot)B(\cdot))(s).$$

In the next section we will show that $I - F_{A,B}$ and $I - G_{A,B}$ are invertible.

3 The Main Results

The following two theorems have already been proved in the literature for the semigroup case and the proofs are adaptable to the non-autonomous case. Since here we generalize these theorems, we shall only sketch the proofs at the points were they mimic those of the semigroup case. See [4] and [10] for more details. On the other hand some points require more care, in particular the generation conditions. These points will accordingly be more detailed.

Theorem 3.1 *Let* (A,U,Z) *be right-compatible and let* $B \in C([0,T], \mathcal{B}_s(X,Z))$*. Then there is a unique evolution family* $V = \{V(t,s) : (s,t) \in \Delta_T\}$ *satisfying*

$$V(t,s) = U(t,s) + (F_{A,B}V)(t,s)$$

for $(s,t) \in \Delta_T$*. Moreover,* $A(I+B)$ *weakly generates* V*, if* A *weakly generates* U*.*

Proof: Consider for $(s,t) \in \Delta_T$ the mappings $F_{A,s,t} \circ B$. By the continuity of γ and $\gamma(0) = 0$, there exists an $\epsilon > 0$, such that $\gamma(\epsilon) < \|B\|_\infty^{-1}$. We find that for all $(s,t) \in \Delta_T$ with $|t-s| < \epsilon$ we have (by monotoneity of γ) $\|F_{A,s,t} \circ B\| < 1$ and thus

$$V(\cdot, s) := (I - F_{A,s,t} \circ B)^{-1} U(\cdot, s)$$

exists in $C([s,t], \mathcal{B}_s(X))$. Using (2) we can patch these solutions together to obtain a function $V \in C(\Delta_T, \mathcal{B}_s(X))$ for which we obtain the formula

$$V(t,s) = U(t,s) + (F_{A,B} V)(t,s),$$

which uniquely defines V and is now used to prove the Kolmogorov equation. Towards this end fix $r \in [0,T]$ and define the family $\widehat{V} \in C(\Delta_T, \mathcal{B}_s(X))$.

$$\widehat{V}(t,s) = \begin{cases} V(t,r)V(r,s) & r \in (s,t) \\ V(t,s) & r \notin (s,t) \end{cases}$$

for $(s,t) \in \Delta_T$. Fix $s \in [0,T]$ and let $W_s, W_r \in D^Z$. Now define

$$\widehat{W}(t,s) = \begin{cases} W_r(t)V(r,s) & r \in (s,t) \\ W_s(t) & r \notin (s,t) \end{cases}$$

We immediately see

$$(F_{A,B}\widehat{W})(t,s) = U(t,r)(F_{A,B}\widehat{W})(r,s) + (F_{A,B}\widehat{W})(t,r)V(r,s).$$

Since $\widehat{W}$ was a construction from a dense set, the same equation holds for $\widehat{V}$. That implies

$$\widehat{V}(t,s) = U(t,s) + (F_{A,B}\widehat{V})(t,s)$$

By uniqueness of V, we have $V = \widehat{V}$ and this family is therefore indeed an evolution family.

We now establish the generation conditions. Suppose, U is weakly generated with D^* the set in X^*, such that for $x^* \in D^*$ the derivative $\partial_1 < U(t,s)x, x^* >$ exists and is continuous. Thus for $x^* \in D^*$:

$$\begin{aligned} < V(t+h,s)x - V(t,s)x, x^* > &= \int_t^{t+h} < U(r,s)x, A(r)^* x^* > dr \\ &+ \int_t^{t+h} < B(r)V(r,s)x, A(r)^* U(t+h,r)^* x^* > dr \\ &+ < (F_{\mathcal{A},\mathcal{B}} V)(t,s), U(t+h,t)x^* - x^* > \end{aligned}$$

By continuity of the integrand we find that $h^{-1} < V(t+h,s)x - x, x^* >$ converges to

$$< U(t,s)x + B(t)V(t,s)x + (F_{A,B}V)(t,s), A(t)^* x^* >$$

as $h \to 0$. Thus $\partial_1^+ < V(t,s)x, x^* > = < (I + B(t))V(t,s)x, A(t)^* x^* >$. This derivative is continuous as well. For the derivative with respect to the second argument, we may apply the same argument for $t = s$. For $s \neq t$, we note that

$$\frac{1}{h}(V(t,s-h) - V(t,s)) = V(t,s)\frac{1}{h}(V(s,s-h) - I)$$

and V is strongly continuous. □

Observe that this theorem applies in cases, where the intersection of the various domains of $A(\cdot)$ may intersect non-densely. Even the domains of the adjoints do not have to be dense. Let $0 = t_0 < \ldots < t_n = 1$ be a partition of $[0,1]$ and $b_i \in [0,1]$, $(0 \leq i < n)$ with $b_i \neq b_{i+1}$ for $0 \leq i < n$. Consider $X = L_p([0,1])$, $(1 \leq p < \infty)$, $A(t)f = f'$ $(f \in D(A(t)))$, and

$$D(A(t)) = W^{1,p}([0,1]) \cap \{f(1) = b_i \int_0^1 f(x)dx\},$$

for $t \in [t_i, t_{i+1})$, $(0 \leq i < n)$. Then

$$D(A(t)) \cap D(A(s)) = \{f \in W^{1,p}([0,1]) : f(1) = \int_0^1 f(x)dx = 0\}$$

if s and t are not in the same subinterval $[t_i, t_{i+1})$. This set is not dense and if we look for physically meaningful solutions of such a problem, they will be non-negative, but no non-trivial positive function exists in that intersection. Such boundary conditions may for instance be a birth rate requirement in mathematical biology. The evolution family on Δ_1 strongly generated by $(A(t))_{t\in[0,1)}$ is

$$U(t,s) = T_j(t-t_j)T_{j-1}(t_j - t_{j-1}) \cdots T_i(t_{i+1} - s),$$

if $t \in [t_j, t_{j+1})$, $s \in [t_i, t_{i+1})$, and $i < j$ (if $i = j$, $U(t,s) = T_i(t-s)$), where

$$(T_j(s)f)(x) = \begin{cases} f(x+s) & x+s \in [0,1) \\ a_j(\int_0^1 e^{b_j t} f(r)dr - \int_0^{b_i t} e^{a_i t - r} f(r)dr) & x-1+s \in [0,1] \end{cases}$$

We have $X^* = L_q([0,1])$, $(1/p + 1/q = 1)$ and with $D = C_0^\infty([0,1])$ in X^* the family U is weakly generated. A reasonable perturbation in this context would be $f(x) \mapsto m(x)f(x)$, where m represents the mortality rate. Cf. [5] for a treatment of this example.

Other evolution families, where the various domains are not identical were presented and studied by H. Amann (cf. [1] for a survey), G. E. Parker [16], and by A. Pazy [17].

Theorem 3.2 *Let (A, U, Z) be left-compatible and $B \in C([0,T], \mathcal{B}_s(Z,X))$. Then there exists a unique evolution family $V = \{V(s,t) : (s,t) \in \Delta\}$ satisfying*

$$V(t,s) = U(t,s) + (G_{A,B}V)(t,s)$$

for $(s,t) \in \Delta$. Moreover, $(I+B)A$ quasi-generates V, if A quasi-generates U.

Proof: The construction of the evolution family is the same as in Theorem 3.1, except that $F_{A,B}$ is replaced by $G_{A,B}$. The generation part is handled differently.

Suppose now that U is quasi-generated by A and let $C(t)$ denote the extension of $A(t)$ such that $\partial_2 U(t,s)x = C(t)U(t,s)x$ for all $x \in D(A(s))$. Let

$C_p(t)x := \lim_{h\to 0} h^{-1}(V(t+h,t)x - x)$ with domain equal to the subspace for which this limit exists. Let $x \in D(A(t))$. Then

$$V(t+h,t)x - x = U(t+h,t)x - x + \int_t^{t+h} V(t+h,r)B(r)C(r)U(r,t)x\,dr$$

and thus $h^{-1}(V(t+h,t)x - x)$ converges to $(I + B(t))A(t)x$ as $h \to 0$. Hence $D(A(t)) \subset D(C_p(t))$. Let $X + \mathcal{B}(X)$ denote the space of affin linear mappings in X, normed by $\|f\|_A = |f(0)| + \|f - f(0)\|_{\mathcal{B}(X)}$. We now as a first step consider for fixed $0 \le s_1 \le s_0 < t_0 \le T$ and $x \in D(A(s_1))$ the map G_{x,s_1,s_0,t_0} defined by

$$(G^{(1)}_{x,s_1,s_0,t_0}W)(t) := (I + B(t))C(t)U(t,s_1)x - (G_{A,B,s_0,t_0}W)(t,s_0)U(s_0,s_1),$$

which maps $C([s_0,t_0], X + \mathcal{B}_s(X))$ into itself. We have, as a matter of course, extended G_{A,B,s_0,t_0} to constant functions in a straight-forward manner. If we choose $t_0 - s_0$, such that $(t_0 - s_0) + \gamma(t_0 - s_0) < (\|V\|_\infty \|B\|_\infty)^{-1}$, the mapping $G^{(1)}_{x,s_1,s_0,t_0}$ is a contraction. We thus obtain a unique fixed point we call $\widehat{V'}_{x,s_1,s_0,t_0}$.

We proceed to estimate

$$D_h := h^{-1}(V(t+h,s_0)x - V(t,s_0)x) - \widehat{V'}_{x,s_0,s_0,t_0}(t).$$

For arbitrary $\epsilon > 0$ choose $h > 0$, such that $T + h \le t_0$,

$$\|h^{-1}(U(t+h,s_0) - U(t,s_0))x - C(t)U(t,s_0)x\| < \epsilon,$$

and

$$\sup_{r\in[s_0,t+h]} \|V(t+h,r)B(r)C(r)U(r,s)x - B(t)C(t)U(t,s_0)x\| < \epsilon.$$

We then find

$$\begin{aligned} \|D_h\| &\le \|h^{-1}(U(t+h,s_0) - U(t,s_0))x - C(t)U(t,s_0)x\| + \|G_{A,B,s_0,t_0}D_h\| \\ &\quad + \|h^{-1}\int_t^{t+h} V(t+h,r)B(r)C(r)U(r,s_0)x\,dr - B(t)C(t)U(t,s_0)x\| \\ &\le 2\epsilon + \gamma(t_0 - s_0)\|B\|_\infty \|D_h\| \end{aligned}$$

But $t_0 - s_0$ was chosen, such that $\gamma(t_0 - s_0)\|B\|_\infty < 1$, and we find that $\|D_h\| \to 0$ as $h \to 0$. This implies the differentiability of $V(\cdot, s_0))x$ in the interval $[s_0, t_0]$. A similar estimate is true for $-h > 0$.

In the second step we consider for $s_2 \le s_1 < s_0$ the mapping

$$\begin{aligned} (G^{(2)}_{x,s_2,s_1,t_0}W)(t) \;:=\; & (I + B(t))C(t)U(t,s_2)x \\ & -(G^{(1)}_{x,s_2,s_1,s_0}\widehat{V'}_{x,s_2,s_1,s_0})(s_0) \\ & -(G_{A,B,s_2,t_0}W)(t,s_1)U(s_1,s_2), \end{aligned}$$

with $x \in D(A(s_2))$ and $(s_0 - s_1) + \gamma(s_0 - s_1) < (\|V\|_\infty \|B\|_\infty)^{-1}$. We proceed with the same fixed point method as above to obtain differentiability in $[s_1, t_0]$. The repetition process corresponds to a split of the involved integrals into sizable pieces. Induction yields the claim for all of $[0, t_0]$ and $t_0 \in [0, T]$ was arbitrary. Therefore $V(t,s)D(A(s)) \subset D(C_p(t))$ for all $(s,t) \in \Delta_T$ and we have proved the claim □

4 Left- and Right-compatibility after Perturbation

In this section, we obtain results on the stability of left- and right-compatibility, which will reduce the burden of checking these conditions in some examples. We consider the conditions to be stable, if a space Z is compatible with the perturbed evolution family, if it was compatible with the unperturbed. We fix $T \in \mathbb{R}^+$ and consider $U, V \in C(\Delta_T, \mathcal{B}_s(X))$. Suppose $B \in L^\infty([0,T], \mathcal{B}_s(X))$, and let

$$V(t,s)x = U(t,s)x + \int_s^t U(t,r)B(r)V(r,s)x\,ds \tag{3}$$

be satisfied for $x \in X$. By choosing $t - s$ small enough, we find that

$$\begin{aligned} V(t,s)x \quad &= \quad U(t,s)x + \sum_{n=1}^{\infty} \int_s^t U(t,t_1)B(t_1) \cdots \int_s^{t_{n-1}} U(t_{n-1},t_n)B(t_n) \\ &\qquad U(t_n,s)x\,dt_n \cdots dt_1. \end{aligned}$$

This follows from the Neumann-series which solves (3). Using this formula we see

$$U(t,s)x = V(t,s)x - \int_s^t V(t,r)B(r)U(r,s)x\,ds. \tag{4}$$

Proposition 4.1 *Suppose that (A,U,Z) is strongly right-compatible and the family $B \in C([0,T], \mathcal{B}_s(X))$. Then $(A+B,V,Z)$ is strongly right-compatible, where V is the evolution family obtained uniquely from U via the variation of constants formula (3).*

Proof: For arbitrary $\Phi \in D^Z \subset C([0,T], \mathcal{B}_s(X,Z))$, such that $F_A\Phi$ exists in its integral form and $x \in X$ we compute

$$\begin{aligned} \|\int_s^t V(t,r)A(r)\Phi(r)x\,dr\| &\leq \|\int_s^t U(t,r)A(r)\Phi(r)x\,dr\| \\ &\quad + \|\int_s^t \int_r^t U(t,r')B(r')V(r',r)A(r)\Phi(r)x\,dr'\,dr\| \\ &\leq \quad \gamma(t-s)\|\Phi\|_\infty |x| \\ &\quad + \|\int_s^t U(t,r')B(r') \int_s^{r'} V(r',r)A(r)\Phi(r)x\,dr\,dr'\| \end{aligned}$$

$$\leq \quad \gamma(t-s)\|\Phi\|_\infty|x| \\ \qquad + (t-s)\|U\|_\infty\|B\|_\infty \sup_{r\in[s,t]} \| \int_s^r V(r',r)A(r)\Phi(r)x\,dr\|.$$

From this follows for $t-s < (\|V\|_\infty\|B\|_\infty)^{-1}$ that

$$\| \int_s^t V(t,r)A(r)\Phi(r)x\,dr\| \leq \frac{\gamma(t-s)}{1-(t-s)\|U\|_\infty\|B\|_\infty}\|\Phi\|_\infty|x|.$$

Moreover,

$$\| \int_s^t V(t,r)B(r)\Phi(r)x\,dr\| \leq (t-s)\|B\|_\infty\|V\|_\infty\|\Phi\|_\infty|x|.$$

Since Φ was taken from a dense set, we have proved the claim. □

Remark 4.2 Due to the simple nature of the bounded perturbation, we find that the generation results for U apply to V. That is, if U is (weakly) generated by A, then V is (weakly) generated by $A+B$.

The same method of proof using the variation of constants formula and changing the order of integration can be applied to corresponding claims for which we omit the proof. These and the above proposition are summarized in the following theorem.

Theorem 4.3 *Let Z be a Banach space and $B \in C([0,T],\mathcal{B}_s(X))$. Suppose U and V are evolution families satisfying (3) and thus (4).*

(i) If (A,U,Z) is strongly right-compatible, then $(A+B,V,Z)$ is strongly right-compatible.

(ii) If (A,U,Z) is left-compatible and D is a set, which is invariant under V, such that $\Phi(\cdot)A(\cdot)x$ is integrable for $x \in D$ and $\Phi \in D_Z$, then $(A+B,V,Z)$ is left-compatible.

(iii) If (A,U,Z) is strongly right-compatible and $B \in C([0,T],\mathcal{B}_s(X,Z))$, such that $B\Phi \in D_Z$ if $\Phi \in D_Z$, then $(A(I+B),W,Z)$ is strongly right-compatible, where $W = U + F_{A,B}W$.

(iv) If (A,U,Z) is left-compatible, $B \in C([0,T],\mathcal{B}_s(Z,X))$, and D is a set invariant under W, such that $\Phi(\cdot)(I+B(\cdot))A(\cdot)x$ is integrable for $x \in D$ and $\Phi \in D_Z$, then $((I+B)A,W,Z)$ is left-compatible, where $W = U - G_{A,B}W$.

The a posteriori conditions in the above theorem are not that hard to check in many cases. For instance: in the autonomous case, $D = D(A(0))$ may serve in case (ii), as it is left invariant by V, whose generator has the same domain, the integrability is also true.

For the moment assume that we drop the injectivity of the embeddings between Z and X in Definitions 2.1 and 2.6. We see that we actually have to consider iB as perturbation if (A,U,Z) is right-compatible, where $i: Z \to X$ is the embedding but not necessarily injective. We may then consider $Z/\mathrm{Ker}(i)$

instead of Z and the quotient mapping $[i]$ is injective. All the relevant estimates still hold. Suppose (A, U, Z_0) is left-compatible (with $i_0 : X \to Z_0$ injective). If (A, U, Z) is also left-compatible, but $i : X \to Z$ is not injective, we may still consider the perturbation Bi. We define $i' : X \to Z \times Z_0$ with $i'(x) = (i(x), i_0(x))$. Then i' is injective and all the relevant estimates hold. (For an elaboration of this argument in the semigroup case, cf. [9].) Thus, if we drop the injectivity assumption, we can easily deduce the following.

Remark 4.4 If (A, U, Z_k) is strongly right- or left-compatible for $k = 1, 2$, then this is also true for (A, U, M), where M may be a closed subspace, a product, or a quotient space of Z_1 and Z_2. In case of right-compatibility we also have right-compatibility of intersections and in case of left-compatibility we have left-compatibility of sums.

Corollary 4.5 *Assume the situation of Theorem 4.3 (iii). Let $(A, U, \widetilde{Z})$ be strongly right-compatible and $B \in C([0, T], \mathcal{B}_s(X, Z))$. Then $(A(I+B), W, \widetilde{Z})$ is right-compatible, where $W = U + F_{A,B}W$. The same is true mutatis mutandis for the situation in (iv).*

Proof: Suppose (A, U, Z) and $(A, U, \widetilde{Z})$ are both strongly right-compatible or both left-compatible. Let $B \in C([0, T], \mathcal{B}_s(X, Z))$ resp. $B \in C([0, T], \mathcal{B}_s(Z, X))$. Then $(A, U, Z \times \widetilde{Z})$ is right- resp. left-compatible. Now, $\mathcal{B}(X, Z) \subset \mathcal{B}(X, Z \times \widetilde{Z})$ trivially in the first case, while any element of $\mathcal{B}(Z, X)$ can be extended in a canonical way to an element of $\mathcal{B}(Z \times \widetilde{Z}, X)$ in the second. Therefore Theorem 4.3 yields the conclusion for $(A_+, U, Z \times \widetilde{Z})$, where $A_+ = A(I + B)$, resp. $(I+B)A$. We point out that by Remark 4.4 any closed subspace of a space fulfilling right-compatibility also fulfills it and the claim is proven. □

Nevertheless, we will continue to make the injectivity assumption, to avoid the cumbersome addition of the mapping i in all formulas.

5 Evolution Semigroups and Evolution Families

The following approach to evolution families is quite old, but has recently received much attention again (cf. e. g. [8], [12], [14]). An evolution family U on Δ_T can be extended trivially to all of $\{(t, s) \in \mathbb{R}^2 : t \geq s\}$ by setting $U(t, s) = I$ if $t \geq s \geq T$ or $s \leq t \leq 0$ and using the Kolmogorov equation. Generating families are accordingly extended. It is now possible to consider an associated semigroup on some suitable space of functions defined on $\mathbb{R}$; we will only consider

$$F(X) = C_0(\mathbb{R}, X) = (\{f \in C(\mathbb{R}, X) : \lim_{|x|\to\infty} f(x) = 0\}, \|\cdot\|_\infty).$$

Then $\mathcal{T}(t)_{t\geq 0}$ is defined by

$$(\mathcal{T}(t)f)(s) = U(s, s-t)f(s-t).$$

As is easily seen, $\mathcal{T}$ is strongly continuous. Let $F^1(X) = \{f \in F(X) : f' \in F(X)\}$ and let $F^{-1}(X)$ be the completion of $F(X)$ with the norm

$$\|f\|_{-1} = \inf_{g \in F^1(X)} (\|g\|_\infty + \|f - g'\|_\infty). \tag{5}$$

$F^{-1}(X)$ is a space of X-valued distributions. $\mathcal{T}$ restricts to a semigroup on $F^1(X)$ and extends to a semigroup in $F^{-1}(X)$. We will assume that U is generated by $A \in C([0,T], \mathcal{B}_s(Y,X))$, where Y a Banach space dense in X, then the generator of $\mathcal{T}$ equals $(\mathcal{B}f)(s) = f'(s) + A(s)f(s)$ in a proper domain, which includes $F^1(Y)$, cf. [14]. We will assume so in the following.

Theorem 5.1 *If $Y \subset Z$ is densely embedded, then (A,U,Z) is right-compatible, if and only if $(\mathcal{B}, \mathcal{T}, F^1(Z))$ is right-compatible.*

Proof: Assume (A,U,Z) is right-compatible. The space $C_0(\mathbb{R}, Y)$ is dense in $C_0(\mathbb{R}, Z)$ and consequently $C([0,t], F^1(Y))$ is dense in $C([0,t], F^1(Z))$. Now, let $\Phi \in C([0,t], F^1(Y))$. Since the integral in Definition 2.1 is then defined, we do not need to worry about the existence of a norming set D^* in $F^1(Z)^*$, it can be chosen as all of $F^1(Z)^*$. We now calculate (all integrals are strong integrals):

$$\begin{aligned}
&\sup_{r \geq 0} \|(\int_0^t \mathcal{T}(s)\mathcal{B}\Phi(s)\, ds)(r)\| \\
&\quad \leq \sup_{r \geq 0} \|\int_0^t U(r, r-s)\Phi(s)'(r-s) ds\| \\
&\qquad + \sup_{r \geq 0} \|\int_0^t U(r, r-s)A(r-s)\Phi(s)(r-s) ds\| \\
&\quad \leq t\|U\|_\infty \|\Phi(\cdot)'\|_{C([0,t],F(Z))} + \gamma(t)\|\Phi\|_{C([0,t],F(Z))} \\
&\quad \leq c_0(\gamma(t) + t)\|\Phi\|_\infty.
\end{aligned}$$

Thus $(\mathcal{B}, \mathcal{T}, F^1(Z))$ is right-compatible.

Assume now that $(\mathcal{B}, \mathcal{T}, F^1(Z))$ is right-compatible. Define

$$\psi_t(s) = \begin{cases} 1 & \text{for } |s| \in [0,t) \\ (1 + \cos(|s| - t))/2 & \text{for } |s| \in [t, t+\pi) \\ 0 & \text{for } |s| \in [t+\pi, \infty) \end{cases}.$$

For a given $\phi \in C([s,t], Z)$ let $\Phi(r)(r') = \psi_t(r')\phi(r)$ for $r \in [0,t]$, $r' \in \mathbb{R}$. Then $\Phi \in C([s,t], F^1(Z))$ and $\|\Phi\|_\infty \leq 2\|\phi\|_\infty$. If $\phi \in C([s,t], Y)$ we find

$$\begin{aligned}
&\|\int_s^t U(t,r)A(r)\phi(r) dr\| \\
&\quad \leq \|\int_0^{t-s} U(t, t-r)A(t-r)\phi(t-r) dr\| \\
&\quad \leq \|\int_0^{t-s} (\mathcal{T}(r)\mathcal{B}\Phi(r))(t) dr\| + \|\int_0^{t-s} (T(r)\Phi(r)')(t) dr\| \\
&\quad \leq \gamma(t-s)\|\Phi\|_\infty + (t-s)\|T\|_\infty \|\Phi'\|_\infty \\
&\quad \leq c_1(\gamma(t-s) + (t-s))\|\phi\|_\infty
\end{aligned}$$

This proves the claim, since by assumption $C([s,t],Y)$ is a dense space in $C([s,t],Z)$. □

Without proof we state the following lemma. Its proof is very technical and is similar to the autonomous case (cf. [9], Lemma 2.34 and [3], Theorem 2.2).

Lemma 5.2 *If (A,U,Z) is left-compatible, then we can extend $G_{A,s,t}$ in Definition 2.6, such that $\|G_{A,s,t}\Phi\| \leq \gamma(t-s)\|\Phi\|_\infty$ holds for all piecewise continuous Φ, with values in $\mathcal{B}(Z,X)$.*

Theorem 5.3 *If $X \subset Z$ is densely embedded, then (A,U,Z) is left-compatible, if and only if $(\mathcal{B},\mathcal{T},F^{-1}(Z))$ is left-compatible.*

Proof: Assume (A,U,Z) is left-compatible. We remark that $\mathcal{T}$ extends to a strongly continuous semigroup on $F^{-1}(X)$. First

$$\begin{aligned}
&\sup_{r\geq 0}\|(\int_0^t \Phi(s)\mathcal{B}\mathcal{T}(s)g\,ds)(r)\| \\
&\leq \sup_{r\geq 0}\|\int_0^t (\Phi(s)(U(\cdot,\cdot-s)g(\cdot-s))')(r)ds\| \\
&\quad + \sup_{r\geq 0}\|\int_0^t (\Phi(s)A(\cdot)U(\cdot,\cdot-s)g(\cdot-s)\,ds)(r)\| \\
&\leq t\|U\|_\infty\|\Phi\|_\infty\|g\|_\infty + \sup_{r\geq 0}\|\int_0^t (\Phi(s)A(\cdot)U(\cdot,\cdot-s)g(\cdot-s)\,ds)(r)\|.
\end{aligned}$$

We now only consider the second term. We note that $S: f \mapsto \int_{-\infty}^{\cdot} e^{s-\cdot}f(s)ds$, defined for all $f \in C_C(\mathbb{R},Z)$, can be extended to an isomorphism from $F^{-1}(Z)$ into $F(Z)$. It is the inverse of $f \to f+f'$. There exists a $\delta > 0$ (independent of f) and $s_0 \in \mathbb{R}$, such that $|S(f)(s_0)| \geq \delta\|f\|_{F^{-1}(Z)}$. For a given $\Phi_0 \in \mathcal{B}(F^{-1}(Z),F(X))$, $r \in \mathbb{R}$, and $f \in F^{-1}(Z)$, we then define $\phi_0 \in \mathcal{B}(Z,X)$ through an extension of $\phi_0 S(f)(s_0) := (\Phi_0 f)(r)$, such that $\|\phi_0\| \leq \delta^{-1}\|\Phi_0\|$. For a given $\Phi \in C([0,T],\mathcal{B}_s(F^{-1}(Z),F(X)))$, $g \in C([0,T],F^{-1}(Z))$, and $\epsilon > 0$, we thus obtain a function $\phi_\epsilon \in C([0,T],\mathcal{B}_s(Z,X))$, such that $\|\phi_\epsilon(\cdot)S(g(\cdot))(s_0(\cdot)) - (\Phi(\cdot)g(\cdot))(r)\|_\infty < \epsilon$, where s_0 is continuous and $\|\phi\|_\infty \leq 2\delta^{-1}\|\Phi\|_\infty$. With $g(s) = A(\cdot)U(\cdot,\cdot-s)f(\cdot-s)$ we may now estimate the second term from above by estimating the following

$$\begin{aligned}
&\|\int_0^t \phi(s)\int_{-\infty}^{s_0(s)} e^{s-s_0(s)}A(s')U(s',s'-s)f(s'-s)ds'\,ds\| \\
&= \|\int_{-\infty}^{\infty}\int_{[0,t]\cap I(s')} e^{\tau(s,s')}\phi(s)A(s'+s)U(s+s',s')f(s')ds\,ds'\| \\
&= \|\int_{-\infty}^{\infty}\int_{[0,t]} \widetilde{\phi}(s)e^{\tau(s,s')}A(s'+s)U(s+s',s')f(s')ds\,ds'\| \\
&\leq \gamma(t)\int_{-T}^{T}\|f(s')\|ds'\|\widetilde{\phi}\|_\infty \\
&\leq \gamma(t)2T\|f\|_\infty 2\delta^{-1}\|\Phi\|_\infty,
\end{aligned}$$

since $A(s) = 0$ for $s \notin [0, T]$. Note that $\tau(s, s') \leq 0$ for all $s, s' \in \mathbb{R}$. Since $s_0(\cdot)$ can be chosen, such that $[0, t] \cap I(s')$ is the union of a finite number of intervals, $\widetilde{\phi}$ is piecewise continuous, an application of Lemma 5.2 proves the claim.

Assume now that $(\mathcal{B}, \mathcal{T}, F^{-1}(Z))$ is left-compatible. Let $x \in X$ and suppose $\phi \in C([0, t], \mathcal{B}_s(Z, X))$. Then with $(\Phi(r)f)(s) = \phi(r + s)f'(s + r)$ we have $\Phi \in C([0, t], \mathcal{B}_s(F^{-1}(Z), F(X)))$ and $\|\Phi_i\|_\infty \leq \|\phi\|_\infty$.

$$\begin{aligned}
&\| \int_s^t \phi(r)A(r)U(r, s)x\,dr\| \\
&= \quad \| \int_0^{t-s} \phi(r + s)A(r + s)U(r + s, s)x\,dr\| \\
&\leq \quad \| \int_0^{t-s} \phi(r + s)(\mathcal{B}\mathcal{T}(r)\psi_t x)(r + s)\,dr\| \\
&\quad + \| \int_0^{t-s} \Phi(r)(U(\cdot, \cdot - r)\psi_t x)(s)\,dr\| \\
&\leq \quad \gamma(t - s)\|\Phi\|_\infty \| \textstyle\int \psi_t\|_\infty |x| + (t - s)\|U\|_\infty \|\Phi\|_\infty |x| \\
&\leq \quad c_1(\gamma(t - s) + (t - s))\|\phi\|_\infty |x|,
\end{aligned}$$

where we have used that $\|\psi_t\|_\infty < 2 + 2\pi$ if $t - s \leq 1$. This proves the claim. □

The assumption that Y is densely embedded in Z or X in Z is not necessary. We have made it however to reduce computation considerably. If it is not the case, the norming and dense sets for (A, U, Z) have to be mapped to norming and dense sets for $(\mathcal{B}, \mathcal{T}, F^1(Z))$ resp. $(\mathcal{B}, \mathcal{T}, F^{-1}(Z))$ in a more careful fashion.

The condition $A \in C([0, T], \mathcal{B}_s(Y, X))$ is technical in nature. It ensures the existence of dense sets. Other conditions are conceivable to ensure such existence. The following lemma provides a starting point in that direction.

Lemma 5.4 *Let X be a Banach spaces. If D is a norming subspace in X^*, then for any compact interval I the space $L_D := \{f \in L_1(I, X^*) : \mathrm{Im(f)} \subset D\}$ is norming for $C(I, X)$.*

Proof: Assume the claim is false and let $f \in C(I, X)$, such that there exists an $\alpha \in [0, 1)$ and for all $g \in L_D$ with $\|g\|_1 \leq 1$

$$\int_I | < f(s), g(s) > | ds \leq \alpha \|f\|_\infty.$$

In particular, choose $g_{t,\epsilon} = (2\epsilon)^{-1}\chi_{|\cdot - t| < \epsilon}$, if $(t - \epsilon, t + \epsilon) \subset I$. Then $\|g_{t,\epsilon}x^*\|_1 = |x^*|$ and $g_{t,\epsilon}x^* \in L_D$, if $x^* \in D$.

$$\int_I < f(s), g_{t,\epsilon}(s)x^* > dt \to < f(t), x^* > \qquad (\epsilon \to 0).$$

Since D is norming, this implies $\|f(t)\| \leq \alpha\|f\|_\infty$ for all $t \in I$, which is a contradiction. □

6 Examples

In this section we will deal with applications of the theory to certain non-autonomous problems. Previously in [3], [4], and [10] the theory was applied to population dynamics, delay equations, (non-linear) lower order terms in elliptic boundary value problems and operators with (non-linear) boundary values. Here we shall apply our theory to potentials in the heat equation and non-autonomous elliptic boundary value problems. Note that the results of this paper may be applied to the examples from the papers cited, as well as this example is subject to the results of those papers. First, consider the following simple problem.

Set $L_{\infty,0}(\mathbb{R}_0^+) := \{f \in L_\infty(\mathbb{R}_0^+) : \lim_{r\to\infty} \text{ess} - \sup_{x>r}|f(x)| = 0\}$. Let α be strictly monotone and absolutely continuous, i. e. $\alpha' \in L_1(\mathbb{R}_0^+)$ and $\alpha'(t) > 0$ for almost all $t > 0$. The evolution family on $C_0(\mathbb{R}_0^+)$ given by

$$(U(t,s)f)(x) = f(x + \alpha(t) - \alpha(s)),$$

$(s,t) \in \Delta_\infty$, $f \in C_0(\mathbb{R}_0^+)$ is strongly continuous. It is generated by $A(t) = \alpha'(t)d/dx$ with maximal domain $C_0^1(\mathbb{R}_0^+)$, if $\alpha' \in C(\mathbb{R}_0^+)$. We also note that with $A_0 := d/dx$ with domain $D(A_0) = C_0^1(\mathbb{R}_0^+)$ in $C_0(\mathbb{R}_0^+)$, the operator $A_0 - \epsilon I$ is invertible for $\epsilon > 0$. More specifically,

$$(A_0 - \epsilon I)^{-1} f(x) = \int_0^\infty e^{-\epsilon s} f(x+s)\, ds,$$

and A_0 generates the translation semigroup $T_0(t)f := f(\cdot - t)$. Consider a function $B \in C(\mathbb{R}_0^+, \mathcal{B}(C_0(\mathbb{R}_0^+), L_{\infty,0}(\mathbb{R}_0^+)))$. (A continuous family of non-continuous multiplications, for instance.)

We find $A(t)f + B(t)f = A(t)(I + R(t)B(t))f - R(t)B(t)f \in L_{\infty,0}(\mathbb{R}_0^+)$ for $f \in D(A_0)$. The operator $R(t)$ is the extension of $(A(t) - I)^{-1} = (\alpha'(t))^{-1}(A - (\alpha'(t))^{-1}I)^{-1}$ to $L_{\infty,0}(\mathbb{R}_0^+)$. Note, that it maps $L_{\infty,0}(\mathbb{R}_0^+)$ into $C_0(\mathbb{R}_0^+)$. We will now prove that $(A, U, \overline{R(0)L_{\infty,0}(\mathbb{R}_0^+)})$ is (weakly) right-compatible. From this follows that an evolution family exists, which satisfies the variation of constant formula that is weakly generated by $A + B$, if $\alpha' \in C(\mathbb{R}_0^+)$

Let $f \in C([0,T], L_{\infty,0}(\mathbb{R}_0^+))$. We have

$$A(t)(A_0 - I)^{-1} = \alpha'(t) + \alpha'(t)(A_0 - I)^{-1}.$$

We then calculate

$$\begin{aligned}
&\|F_{A,s,t}(A_0 - I)^{-1} f\| \\
&= \sup_{x\geq 0} \left| \int_s^t U(t,r)A(r)((A_0 - I)^{-1} f)(r,\cdot)(x)dr \right| \\
&\leq \sup_{x\in\mathbb{R}} \left| \int_s^t \alpha'(r) f(r, x - \alpha(t) + \alpha(r))dr \right| \\
&\quad + \sup_{x\in\mathbb{R}} \left| \int_s^t \alpha'(r)((A_0 - I)^{-1} f)(r, x - \alpha(t) + \alpha(r))dr \right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_s^t |\alpha'(r)| \sup_{\tau\in[0,T]} (1 + \|(A_0 - I)^{-1}\|)\|f(\tau,\cdot)\|_\infty dr \\
&\leq \gamma(t-s)\|f\|_{C([0,t],L_{\infty,0}(\mathbb{R}_0^+))} \\
&\leq \gamma(t-s)M\|R(0)f\|_{C([0,t],R(0)L_{\infty,0}(\mathbb{R}_0^+))}.
\end{aligned}
$$

To prove that

$$
\int_s^t U(t,r)A(r)(A_0 - I)^{-1} f(r,\cdot)dr \in C_0(\mathbb{R}_0^+),
$$

i. e. it has zero limits as $|x| \to \infty$ essentially mimics the argument just given. In the calculations made, we have used the fact, that $A(t)$, $(t \in [0,T])$ extends to a continuous operator in $\mathcal{B}(C_0(\mathbb{R}_0^+), C_0^{-1}(\mathbb{R}_0^+))$. ($C_0^{-1}(\mathbb{R}_0^+)$ is the completion of $C_0(\mathbb{R}_0^+)$ with norm given in (5).) In $C_0^{-1}(\mathbb{R}_0^+)$ the family U is also continuous and the calculated bounds hold. Of course, we could have made all estimates in the "weak" form by choosing $D = C_C^\infty(\mathbb{R}_0^+) \subset \cap\{D(A(t)^* : t \in [0,T]\}$ as norming space in the dual of $C_0(\mathbb{R}_0^+)$. This proves the claim on right-compatibility. By Theorem 3.1, we know that $A + B$ weakly generates a (strong) evolution family in $C_0(\mathbb{R}_0^+)$ on Δ_T, if $\alpha' \in C(\mathbb{R}_0^+)$.

Another construction, which features non-constant domains in $C([0,1])$, is given by $A(t) = d/dx$, $D(A(t)) = \{f \in C^1[0,1], f'(0) = \alpha(t)f(0)\}$ for some $\alpha \in C(\mathbb{R}_0^+)$. The evolution family strongly generated by A on Δ_1 is given by

$$
(U(t,s)f)(x) = \begin{cases} f(x-t+s) & x-t+s \in [0,1] \\ f(0) - \int_s^{t-x} \alpha(r)f(r-s)dr & x-t+s \notin [0,1] \end{cases}.
$$

A space which is compatible with U and A is the space $L_\infty([0,1])$. The calculations require only a little more effort than in the previous example.

Similar operators and especially extrapolations spaces have been studied extensively (e. g. [7], [15]).

As was seen in the previous example, $A(\cdot)$ is not required to fulfill restrictive time-regularity conditions for the two main theorems to hold. The next example will show that the conditions on $B(\cdot)$ are also sufficiently general to obtain evolution families in the parabolic case for generators not Hölder continuous. Recall that parabolic problems have smoothing properties that are not present in the general case. Also recall the space $C^\alpha(I,X) = \{f \in C(I,X) : \|f(t) - f(s)\| \leq C|t-s|^\alpha; s,t \in I\}$ for any compact interval I and Banach space X.

We take $A(t)_{t\in[0,T]}$ to be a family of generators of analytic semigroups satisfying

$$
\sup_{t\in[0,T]} \|(\lambda I - A(t))^{-1}\| \leq M/(1 + |\lambda|)
$$

for $\Re(\lambda) > 0$. In particular $(0,\infty) \subset \rho(A(t))$ for all $t \in [0,T]$. We introduce the spaces $X(t)_k = D(A(t)^k)$ with graph norms and $|k| \in \mathbb{N}_0$. This constitutes the so-called abstract Sobolev towers for $A(t)$ (cf. [13], [21]). Then by using an admissible interpolation functor $\{(\cdot,\cdot)_\theta\}_{0<\theta<1}$, we obtain a scale $\{X(t)_r\}_{r\in\mathbb{R}}$

by setting $X(t)_r = (X(t)_{[r]}, X(t)_{[r]+1})_{r-[r]}$ for $r \in \mathbb{R}$, $|r| \notin \mathbb{N}_0$ ([r] denotes the largest integer less than or equal to r.) Let Λ be the set of all $\alpha \in \mathbb{R}$, such that

$$X(t)_\alpha = X(0)_\alpha =: X_\alpha$$

for all $t \in [0,T]$ and that the norms in these spaces are uniformly equivalent, e. g. $0 \in \Lambda$. The r-realization of $A(t)$ is called $A(t)_r$; it is the (continuous) restriction/extension/interpolation of $A(t) : X(t)_{r+1} \to X(t)_r$. Assume that $\alpha, 1+\alpha \in \Lambda$, $-1 < \alpha < 0$, and $A_\alpha := A(\cdot)_\alpha \in C^s([0,T], \mathcal{B}(X_{\alpha+1}, X_\alpha))$ with $-\alpha < s < 1$. We also consider A_α as operator family in X_α, where it consists of closed operators. Then $A_0(\cdot)$ generates an evolution family in X. Cf. [2] for details on this method. Also several properties of the evolution family generated by $A(\cdot)$, which we need and state without source may be found there.

Theorem 6.1 *Let $B \in C([0,T], \mathcal{B}(X_0, X_\alpha))$, such that*

$$A_\alpha(\cdot)B(\cdot)A_0(\cdot)^{-1} \in C([0,T], \mathcal{B}(X_0, X_\alpha)).$$

Then (the 0-realization of) $A(\cdot) + B(\cdot)$ generates an evolution family in X.

Proof: First, we recall that the evolution family $U_\alpha(\cdot,\cdot)$ (generated by $A_\alpha(\cdot)$) satisfies the estimate

$$(t-s)^{\epsilon-\alpha} \|U_\alpha(t,s)\|_{\mathcal{B}(X_\alpha, X_0)} \le C_1(\epsilon) \tag{6}$$

for $(s,t) \in \Delta_T$ and for $0 < \epsilon < 1+\alpha$. This can be seen by interpolation. From this we conclude that

$$\| \int_s^t \Phi(r)B(r)U_\alpha(r,s)x dr\| \le \|\Phi\|_\infty \|B\|_\infty C_2(\epsilon)(t-s)^{\alpha-\epsilon+1}\|x\|,$$

which shows by Theorem 3.2 that $A_\alpha + B = (A+B)_\alpha$ quasi-generates an evolution family $V_\alpha(\cdot,\cdot)$ in X_α.

Now choose $\lambda > 0$, such that

$$\sup_{t\in[0,T]} (\|(\lambda I - A_\alpha(t))^{-1}\|_{\mathcal{B}(X_\alpha, X_0)} + \|(\lambda I - A_0(t))^{-1}\|_{\mathcal{B}(X_0, X_{\alpha+1})}) < \|B\|_\infty^{-1}.$$

Then we deduce from the formulas

$$\begin{aligned}(\lambda I - (A+B)_\alpha(t))^{-1} &= (\lambda I - A_\alpha(t))^{-1}(I - B(t)(\lambda I - A_\alpha(t))^{-1})^{-1}\\ (\lambda I - (A+B)_0(t))^{-1} &= (I - (\lambda I - A_\alpha(t))^{-1}B(t))^{-1}(\lambda I - A_0(t))^{-1}\\ (\lambda I - (A+B)_0(t))^{-1} &= (\lambda I - A_0(t))^{-1}(I - B(t)(\lambda I - A_\alpha(t))^{-1})^{-1},\end{aligned} \tag{7}$$

where the last equation follows from the first two, that $\lambda I - (A+B)_0$ is invertible. We know that the following is true for the evolution family generated by A_0:

$$(A_0(t)U(t,s)A_0^{-1}(s))_{(s,t)\in\Delta_T} \in C(\Delta_T, \mathcal{B}(X_0)). \tag{8}$$

Even more is true. For $(s,t) \in \Delta_T$ we have

$$(t-s)^{\alpha-\epsilon}\|A_\alpha(t)U_\alpha(t,s)A_\alpha(s)^{-1}\|_{\mathcal{B}(X_\alpha, X_0)} \le C_3(\epsilon). \tag{9}$$

by possibly adjusting $\epsilon > 0$. We now define g by

$$g(t,s) = (\lambda I - (A+B)_0(t))U(t,s)(\lambda I - (A+B)_0(s))^{-1}.$$

From (7), (8) we obtain that $g \in C(\Delta_T, \mathcal{B}(X_0))$.

We then define $\mathcal{F}$ through

$$\mathcal{F}(\Phi)(t,s) = g(t,s) + \int_s^t \Phi(r)(\lambda I - (A+B)(r))B(r)(\lambda I - (A+B)(r))^{-1} g(r,s)dr,$$

where Φ is in a suitable function space to be determined. We first calculate an estimate for part of the integrand.

$$\begin{aligned}
&(\lambda I - (A+B)(r))B(r)(\lambda I - (A+B)(r))^{-1} \\
&= (I - B(r)(\lambda I - A(r))^{-1})(\lambda A(r)^{-1} - I)A(r)B(r)A(r)^{-1} \cdot \\
&\quad \cdot (I - (\lambda I - A(r))^{-1}B(r))^{-1}(\lambda A(r)^{-1} - I)^{-1}
\end{aligned}$$

From this follows that $(\lambda I - (A+B)(\cdot))B(\cdot)(\lambda I - (A+B)(\cdot))^{-1}$ has a realization as continuous mapping in $\mathcal{B}(X_0, X_\alpha)$. We can now use the Hölder estimate (9) to see that $\mathcal{F}$ is well-defined and continuous, if considered as mapping

$$\mathcal{F} : C(\Delta_{s_0 t_0}, \mathcal{B}(X)) \to C(\Delta_{s_0 t_0}, \mathcal{B}(X))$$

for any $(s_0, t_0) \in \Delta_T$. Moreover, $\mathcal{F}$ is a contraction when $\delta := t_0 - s_0$ is chosen sufficiently small. Its fixed point we call $\widehat{V}$. The family $\widehat{V}$ can be extended to all of Δ_T in the usual way, since the contraction property of $\mathcal{F}$ does not depend on the particular choice of (s_0, t_0), but only on δ. By uniqueness we identify $\widehat{V}$ with this extension.

Straightforward calculations using the uniqueness of $\widehat{V}$ now yields

$$\widehat{V}(t,s) = (\lambda I - (A+B)_0(t))V_\alpha(t,s)(\lambda I - (A+B)_0(s))^{-1}.$$

From the quasi-generation on V_α in X_α, we infer that $V_\alpha(t,s)$ actually maps $D((A+B)_0(s))$ into $D((A+B)_0(t))$ and $\partial_1 V_\alpha(t,s)x = (A+B)_0(t)V_\alpha(t,s)x$ for $x \in D((A+B)_0(s))$. In other words, $(A+B)(\cdot)$ generates an evolution family in X. □

Note, that although we have sub-generating conditions on V_α (in X_α), we have full generation in X.

Consider the heat equation with a potential in $L_p(\mathbb{R}^n)$ $(1 \le p < \infty)$:

$$\partial_t u(x,t) = \Delta u(x,t) + V(x,t)u(x,t).$$

The semigroup generated by the Laplacian

$$\Delta = \sum_{j=0}^{n} \frac{\partial^2}{\partial x_j^2}$$

is given by a convolution integral $T(t)f = k_t * f$, where the heat kernel is $k_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$ with $\|k_t\|_1 = 1$. Thus T is a contraction semigroup in $L_p(\mathbb{R}^n)$ $(1 \leq p < \infty)$. When considering

$$T(t) : L_q(\mathbb{R}^n) \to L_p(\mathbb{R}^n)$$

we obtain $\|T(t)\|_{\mathcal{B}(L_q(\mathbb{R}^n), L_p(\mathbb{R}^n))} = \|k_t\|_r = r^{-n/2}(4\pi t)^{-(1/p-1/q)n/2}$ with $1 + 1/p = 1/q + 1/r$. For arbitrary $\Phi \in C([0,T], \mathcal{B}_s(L_p(\mathbb{R}^n), L_q(\mathbb{R}^n))$ we have

$$\| \int_a^b T(b-c)\Phi(c)f \, dc\|_p \leq \widetilde{C}(p,q,n)\|\Phi\|_\infty \|f\|_p \int_a^b c^{-n/2s} dc,$$

$1/p = 1/s + 1/q$. Thus, if we have $2s > n$, the latter integral exists, and with the inequality $b^\epsilon - a^\epsilon \leq (b-a)^\epsilon$ for $0 < \epsilon < 1$ we obtain

$$\| \int_a^b T(b-c)\Phi(c) \, dc\| \leq C(p,q,n)\|\Phi\|_\infty (b-a)^{1-n/2s}.$$

Used together with a Sobolev imbedding theorem, this proves the strong right-compatibility of $(\Delta, T, (\epsilon I - \Delta)^{-1} L_q(\mathbb{R}^n))$, where T is considered in $L_p(\mathbb{R}^n)$. Moreover, if $V \in C([0,T], L_s(\mathbb{R}^n))$ we have

$$V(t) : L_p(\mathbb{R}^n) \to L_q(\mathbb{R}^n)$$

and $V \in C([0,T], \mathcal{B}_s(L_p(\mathbb{R}^n), L_q(\mathbb{R}^n)))$. In particular $\Delta + V$ weakly generates an evolution family in $L_p(\mathbb{R}^n)$, if $V \in C([0,T], L_s(\mathbb{R}^n))$ with $2s > n$ and $p \geq s/(s-1)$. From [5] we infer that $\Delta + V$ strongly generates in $L_p(\mathbb{R}^n)$, if $V \in C^1([0,T], L_s(\mathbb{R}^n))$. Using the same estimates one can show that $(\Delta, T, (\epsilon I - \Delta)^{-1} L_p(\mathbb{R}^n))$ is left-compatible, where where $T(\cdot)$ is considered in $L_q(\mathbb{R}^n)$. Therefore $\Delta + V$ quasi-generates an evolution family in $L_q(\mathbb{R}^n)$ for $q > s$ and $2s > n$, if $V \in C([0,T], L_s(\mathbb{R}^n))$.

Similar results were recently obtained by F. Räbiger, A. Rhandi, and R. Schnaubelt [20].

References

[1] H. Amann. Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems. In *Function Spaces, Differential Operators and Nonlinear Analysis*, Teubner Texte zur Mathematik, pages 1–126, Stuttgart, 1993. Teubner.

[2] H. Amann. *Linear and Quasilinear Parabolic Problems.* Birkhäuser, Basel, 1995.

[3] W. Desch, I. Lasieka, and W. Schappacher. Feedback boundary control problems for linear semigroups. *Isr. J. Math.*, 51:177–207, 1989.

[4] W. Desch and W. Schappacher. Some generation results for perturbed semigroups. In P. Clément, G. Lumer, S. Invernizzi, E. Mitidieri, and I. I. Vrabie, editors, *Semigroup Theory and Applications*, volume 116 of *Lecture Notes in Pure and Applied Mathematics*, pages 125–152, New York, 1987. Marcel Dekker.

[5] W. Desch, W. Schappacher, and K. P. Zhang. Semilinear evolution equations,. *Houston J. Math.*, 15:527–552, 1989.

[6] K.-J. Engel, R. Nagel, F. Räbiger, and U. Schlotterbeck, editors. *Tübinger Berichte zur Funktionalanalysis*, Tübingen, 1993/1994. Eberhard-Karls-Universität Tübingen.

[7] T. Graser and P. Mezo. Operator multipliers, strongly continuous semigroups and abstract extrapolation spaces. In Engel et al. [6], pages 63–80.

[8] J. S. Howland. Stationary scattering theory for time-dependent hamiltonians. *Math. Ann.*, 207:315–335, 1974.

[9] M. Jung. *Multiplikative Störungen von C_0-Halbgruppen.* PhD thesis, Technische Universität Berlin, Berlin, Februar 1994.

[10] M. Jung. Multiplicative perturbations in semigroup theory with the (Z)-condition. *Semigroup Forum*, 52:197–211, 1996.

[11] M. Jung. Evolution families and generators of semigroups. *Houston J. Math.*, 23:357–383, 1997.

[12] Y. Latushkin and S. Montgomery-Smith. Evolutionary semigroups and Lyapunov theorems in Banach spaces. *J. Funct. Anal.*, 127:173–197, 1995.

[13] R. Nagel, editor. *Semigroups of Positive Operators.* Lecture Notes in Mathematics. Springer, Berlin, 1986.

[14] R. Nagel. Semigroup methods for nonautonomous cauchy problems. In G. Ferreyra, G. R. Goldstein, and F. Neubrander, editors, *Evolution Equations*, volume 168 of *Lecture Notes in Pure and Applied Mathematics*, pages 301–316, New York, 1992. Marcel Dekker.

[15] R. Nagel, G. Nickel, and S. Romanelli. Identification of extrapolation spaces for unbounded operators. In Engel et al. [6], pages 137–152.

[16] G. E. Parker. Semigroup structure underlying evolutions. *Int. J. Math. Math. Sci.*, 5:31–40, 1982.

[17] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer, Berlin, 1983.

[18] S. Piskarev and S.-Y. Shaw. Multiplicative perturbations of C_0-Semigroups and some applications to step responses and cumulative outputs. *J. Funct. Anal.*, 128:315–340, 1995.

[19] S. Piskarev and S.-Y. Shaw. Perturbation and comparison of cosine operator functions. *Semigroup Forum*, 51:225–246, 1995.

[20] F. Räbiger, A. Rhandi, and R. Schnaubelt. Perturbation and an abstract characterization of evolution semigroups. *J. Math. Anal. Appl.*, 198:516–533, 1996.

[21] T. Walther. *Abstrakte Sobolev-Räume und ihre Anwendung auf die Störungstheorie für Generatoren von C_0-Halbgruppen.* PhD thesis, Eberhard-Karls-Universität Tübingen, Tübingen, 1986.

List of Symbols

O	Oh	0	zero
l	el	1	one
K	uppercase k	k	lowercase k
α	alpha	$\mathcal{A}$	script A
$\mathcal{B}$	script B	$\Box$	box
$\circ$	circle	χ	chi
$\cap$	stylized U open upward	δ	delta
Δ	Delta	ϵ	epsilon
$\mathcal{F}$	script F	$\mathcal{G}$	script G
$\in$	stylized epsilon	$\{$	left braces
∞	infinity	$\int$	integral
λ	lambda	Λ	Lambda
$\mapsto$	barred arrow	$\mathbb{N}$	barred N
$\notin$	slashed $\in$	$\neq$	slashed $=$
$\oplus$	circled plus	ψ	psi
ϕ	phi	Φ	Phi
∂	stylized delta	π	pi
$\mathbb{R}$	barred R	ρ	rho
$\subset$	stylized U open right	$\sum$	Sigma
σ	sigma	τ	tau
$\rightarrow$	arrow	$\times$	cross
$\mathcal{T}$	script T	θ	theta
$\widetilde{x}$	tilde (above x)	$\widehat{x}$	hat (above x)
$\overline{x}$	bar (above x)		