MULTIPLIER THEOREMS ON VECTOR-VALUED BESOV SPACES

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ABSTRACT. We prove generalisations of the Mikhlin and Marcinkiewicz multiplier theorems for vector-valued Besov spaces, using different geometric properties of the underlying vector spaces. We apply the result to Volterra Integral Equations.

1. PRELIMINARIES

We fix our notation and recall the definitions of function spaces we will use as well as some of their important properties. We let $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ and $\mathbb{R}_+^0 = \{x \in \mathbb{R} : x \geq 0\}$. X and Y denote topological vector spaces and $\mathcal{B}(X, Y)$ is the space of continuous linear operators from X into Y. For $F : \mathbb{R}^N \to \mathcal{B}(X, Y)$ and $g : \mathbb{R}^N \to X$ we defined the Y-valued convolution integral as usual by

$$(F * g)(t) = \int_{\mathbb{R}^N} F(t-s)g(s)ds$$

whenever the Bochner integral exists. $\mathcal{S}(\mathbb{R}^N, X)$ is the space of rapidly decreasing X-valued functions. Similarly we define for X-valued functions the Bochner spaces $L_p(\mathbb{R}^N, X)$, and the Sobolev spaces $W_p^n(\mathbb{R}^N, X)$ for $1 \leq p \leq \infty$ and $N \in \mathbb{N}$. The space of X-valued, tempered distributions is given by $\mathcal{S}'(\mathbb{R}^N, X) = \mathcal{B}(\mathcal{S}(\mathbb{R}^N), X)$, cf. [9], Section 3.1 for details.

The (N-dimensional) Fourier transform of f we denote by $\widehat{f}$ or $\mathcal{F}f$, where

$$\widehat{f}(t) = (\mathcal{F}f)(t) = \int_{\mathbb{R}^N} e^{-i<s,t>} f(s)ds$$

for $t \in \mathbb{R}^N$ and $f \in \mathcal{S}(\mathbb{R}^N, X)$. The inverse Fourier transform is known to be

$$(\mathcal{F}^{-1}f)(z) = -(2\pi)^{-N}\mathcal{F}(f(-\cdot))(z). \tag{1}$$

Note that for an Hilbert space X the (extended) mappings $\mathcal{F}$ and $\mathcal{F}^{-1}$ are actually continuous inverses of each other in $L_2(\mathbb{R}^N, X)$, while this is not the case for other spaces we will consider.

A Banach space X is said to have *Fourier type* p for some $1 \leq p \leq 2$, if for p the Hausdorff-Young inequality holds, i. e. there exists a constant $C_{\mathcal{F}}(p) \geq 0$ such that for all $f \in L_p(\mathbb{R}^N, X)$ we have $\|\mathcal{F}f\|_q \leq C_{\mathcal{F}}(p)\|f\|_p$, where $1/q + 1/p = 1$. Obviously, every Banach space has type 1. Moreover, the Banach space X has Fourier type 2 if and only if X is a Hilbert space. A space $L_q(\Omega, \mu)$ has Fourier type $p = \min(q, q/(q-1))$. Also the dual space, every closed subspace and quotient space of a Banach space X has the same Fourier type as X. (Cf. [6].)

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A Banach space X is said to have the *UMD property*, if there exists a $1 < p_0 < \infty$, such that the Hilbert transform $\mathcal{H}$ given by

$$(\mathcal{H}f)(s) = \int_{\mathbb{R}} (t-s)^{-1} f(s) ds$$

extends to an element of $\mathcal{B}(L_{p_0}(\mathbb{R}^N, X))$. Then it is also true that the Hilbert transform bounded on $L_p(\mathbb{R}^N, X))$ for all $1 < p < \infty$. Subspaces, quotient spaces and products of spaces that have UMD also have the UMD property. UMD implies reflexivity, even uniform convexity, but not conversely, cf. [3]. The spaces $L_p(\Omega, \mu)$ have UMD for any measure space (Ω, μ) and $1 < p < \infty$.

Throughout this article we assume sums without index range to have a finite index set. Let I_j^N denote the sets

$$\{x \in \mathbb{R}^N : 2^{j-1} < |x| \leq 2^{j+1}\}$$

for $j \in \mathbb{N}$. Also $I_0^N := \{|x| \leq 2\}$.

We define the *Besov spaces* $B^s_{p,q}(\mathbb{R}^N, X)$ for $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$ as in [9]: let Φ be the set of all sequences $(\phi_j^N)_j$ in $\mathcal{S}(\mathbb{R}^N)$, such that

$$\begin{array}{lll} \text{(i)} & \operatorname{supp}\phi_j^N \subset I_j^N, & j \in \mathbb{N}_0, \\ \text{(ii)} & \sum_{j=0}^{\infty} \phi_j^N(s) = 1, & s \in \mathbb{R}^N, \\ \text{(iii)} & \sup_j \|\mathcal{F}^{-1}\phi_j^N\|_1 < \infty. & \end{array}$$

Then for a fixed sequence from Φ we define the $B^s_{p,q}$-norm

$$\|f\|_{B^s_{p,q}} := (\sum_{j=0}^{\infty} (2^{js} \|\mathcal{F}^{-1}(\phi_j^N) * f\|_p)^q)^{1/q},$$

and the Besov space $B^s_{p,q}(\mathbb{R}^N, X)$ is the set of all distributions in $\mathcal{S}'(\mathbb{R}^N, X)$ which have a finite $B^s_{p,q}$ norm supplied with that norm. Note that different sequences in Φ yield equivalent norms (cf. [10]). To make calculations simpler, we often use a special sequence $(\phi_j^N)_j$ generated by a positive function $\phi \in \mathcal{S}(\mathbb{R})$ with support in $[1/2, 2]$, such that $\sum_{|k|<\infty} \phi(2^{-k}\cdot) \equiv 1$. Then put $\phi_j^N = \phi(|2^{-j}\cdot|)$ for $j \in \mathbb{N}$ and $\phi_0^N = \sum_{k \leq 0} \phi(|2^{-k}\cdot|)$. A well-known (cf. [10]) equivalent norm for the Besov spaces $B^s_{p,q}(\mathbb{R}, X)$ for $0 < s < 1$ is given by

$$\|f\|'_{B^s_{p,q}} = \|f\|_p + \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \|\frac{f(t) - f(s)}{|t-s|^s}\|_p^q |t-s|^{-1} ds dt \right)^{1/q}.$$

Another decomposition of $\mathbb{R}^N$ related to Besov spaces is the Lizorkin decomposition. Let

$$K^N_{k,\tau} := \{x \in \mathbb{R}^N : (\tau_j - 1/2)2^{k-1} < x_j \leq (\tau_j + 1/2)2^{k-1}\}$$

for $\tau \in \Gamma_N := \{\pm\frac{1}{2}, \pm\frac{3}{2}\}^N$ with $\max_j |\tau_j| = 3/2$ and $k \in \mathbb{N}$, and let $K_0 = \{x \in \mathbb{R}^N : |x|_\infty \leq 1\}$.

The *Bessel potential spaces* $H^s_p(\mathbb{R}^N, X)$ for $1 \leq p < \infty$, $s \in \mathbb{R}$ are defined as in [9] as the completion of the set

$$\{f \in \mathcal{S}(\mathbb{R}^N, X) : \mathcal{F}^{-1}[(1 + |\cdot|^2)^{s/2} \mathcal{F}f] \in L_p(\mathbb{R}^N, X)\}.$$

with the norm $\|f\|_{H^s_p} = \|\mathcal{F}^{-1}[(1 + |\cdot|^2)^{s/2}\mathcal{F}f]\|_p$.

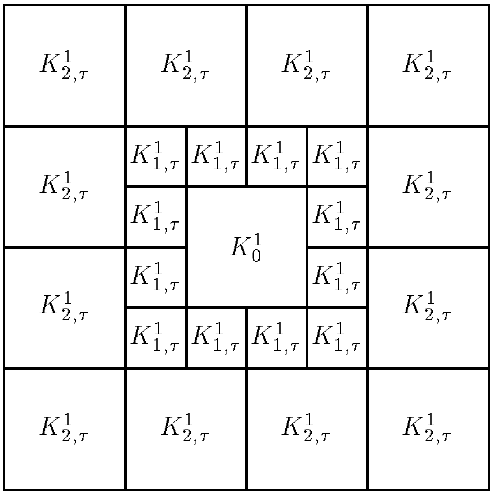

FIGURE 1. $\tau \in \{\pm 1/2, \pm 3/2\}^2$, $|\tau_1|$ or $|\tau_2|$ equal to $3/2$

There are several relations between these space, which we quote here for further reference (cf. e. g. [9]; [11], Section 4.3). All inclusions are understood to be continuous embeddings. Let $1 \le p, q < \infty$, then

$$\begin{array}{rl}
W^n_p(\mathbb{R}^N, X) \subset B^s_{p,q}(\mathbb{R}^N, X) & s < n; \\
B^s_{p,q_0}(\mathbb{R}^N, X) \subset B^s_{p,q_1}(\mathbb{R}^N, X) & 1 \le q_0 \le q_1 \le \infty, s \in \mathbb{R}; \\
B^{s_0}_{p,\infty}(\mathbb{R}^N, X) \subset B^{s_1}_{p,1}(\mathbb{R}^N, X) & s_0 > s_1 \\
H^{s+\epsilon}_p(\mathbb{R}^N, X) \subset B^s_{p,q}(\mathbb{R}^N, X) \subset H^{s-\epsilon}_p(\mathbb{R}^N, X) & s \in \mathbb{R}, q \neq \infty \\
B^s_{p,1}(\mathbb{R}^N, X) \subset H^s_p(\mathbb{R}^N, X) \subset B^s_{p,\infty}(\mathbb{R}^N, X) & s \in \mathbb{R}
\end{array}$$

The last set of inclusions can be proved just as in the scalar-valued analog (compare e. g. the proof [2], Theorem 6.2.4), by applying Remark 2.4 below with $X = \mathbb{C}$ to the uniformly norm-bounded sequence of multipliers $2^{ms}(2^{-2m} + |\cdot|^2)^{s/2}\phi(\cdot)$ $(m \ge -1)$.

If X has the UMD property, then $W^n_p(\mathbb{R}^N, X) = H^n_p(\mathbb{R}^N, X)$ with equivalent norms for all $n \in \mathbb{N}$, while any inclusion $W^n_p(\mathbb{R}^N, X) \subset H^n_p(\mathbb{R}^N, X)$ or $H^n_p(\mathbb{R}^N, X) \subset W^n_p(\mathbb{R}^N, X)$ for odd n implies that X has the UMD property (cf. [11], Section 3.8).

For more information regarding vector-valued Besov, Bessel potential, and Sobolev spaces cf. [7], [9], and [11].

2. Mikhlin type Multipliers

We assume in the following that X and Y are arbitrary Banach spaces and will state any necessary conditions on these spaces in the propositions where needed.

Lemma 2.1. *Let $k : \mathbb{R}^N \to \mathcal{B}(X, Y)$ be a function, such that k and $k(\cdot)^* : \mathbb{R}^N \to \mathcal{B}(Y^*, X^*)$ are strongly measurable and*

$$\int_{\mathbb{R}^N} \|k(s)x\|_Y \, ds \le C_0 \|x\|_X, \qquad \int_{\mathbb{R}^N} \|k(s)^* y^*\|_{X^*} \, ds \le C_1 \|y^*\|_{Y^*} \tag{2}$$

Then $(Kf)(t) = \int_{\mathbb{R}^N} k(t-s) f(s) ds$, $t \in \mathbb{R}^N$, for $f \in \mathcal{S}(\mathbb{R}^N, X)$ can be extended to a bounded operator $K : L_p(\mathbb{R}^N, X) \to L_p(\mathbb{R}^N, Y)$ for $1 \le p \le \infty$.

Proof. Suppose $f \in \mathcal{S}(\mathbb{R}^N, X)$. Then

$$\begin{aligned}\int_{\mathbb{R}^N} \|(Kf)(t)\| dt &\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \|k(t-s)f(s)\| dt\, ds \\ &\leq C_0 \int_{\mathbb{R}^N} \|f(s)\| ds \leq C_0 \|f\|_1.\end{aligned}$$

Thus there exists an extension $K : L_1(\mathbb{R}^N, X) \to L_1(\mathbb{R}^N, Y)$. If $p = \infty$ we deal with $(Kf)(t) = \int_{\mathbb{R}^N} k(t-s)f(s)ds$ for $f \in L_\infty(\mathbb{R}^N, X)$ as a Dunford-Pettis integral and estimate for $y^* \in Y^*$.

$$\begin{aligned}| < y^*, (Kf)(t) > | &\leq \int_{\mathbb{R}^N} | < k(t-s)^* y^*, f(s) > | ds \\ &\leq \int_{\mathbb{R}^N} \|k(t-s)^* y^*\| \, \|f(s)\| ds \leq C_1 \|y^*\| \, \|f\|_\infty.\end{aligned}$$

Thus there exists an extension $K : L_\infty(\mathbb{R}^N, X) \to L_\infty(\mathbb{R}^N, Y)$. If $L^0_\infty(\mathbb{R}^N, X)$ denotes the closure of simple functions $\sum x_k \chi_{A_k}$ with $x_k \in X$ and $\mathrm{vol}(A_k) < \infty$, then one can check that K maps $L^0_\infty(\mathbb{R}^N, X)$ into $L^0_\infty(\mathbb{R}^N, X)$. Indeed, for $f = x\chi_A$ we have $(Kf)(t) = \int_{t-A} k(s)x\, ds \to 0$ for $|t| \to \infty$. The Riesz-Thorin Theorem (cf. [2], Theorem 5.1.2) now yields the claim for $1 < p < \infty$. □

Remark 2.2. Lemma 2.1 is an extension of [4], Theorem 1.14. It is shown there that the second condition cannot be omitted. For Hilbert spaces Lemma 2.1 is also implicit in the proof of [6], Corollary 10.4.

Proposition 2.3. *Let X, Y have Fourier type $p \in [1, 2]$ and for some $s > N/p$ let $m \in H^s_p(\mathbb{R}^N, \mathcal{B}(X, Y))$, then*

$$(\widehat{Kf}(t) = m(t)\hat{f}(t), \quad t \in \mathbb{R}^N,$$

for $f \in \mathcal{S}(\mathbb{R}^N, X)$ can be extended to a bounded operator $K : L_q(\mathbb{R}^N, X) \to L_q(\mathbb{R}^N, Y)$ for all $1 \leq q \leq \infty$ and $\|K\| \leq C\|m\|_{H^s_p}$, where C is independent of q and m.

Proof. Let $1/p + 1/p' = 1$. We then get for $g \in H^s_p(\mathbb{R}^N, Y)$

$$\begin{aligned}(2\pi)^N \int_{\mathbb{R}^N} \|(\mathcal{F}^{-1} f)(t)\| dt &= \int_{\mathbb{R}^N} \|\hat{g}(t)\| dt \\ &\leq \left(\int_{\mathbb{R}^N} (\|\hat{g}(t)\|(1+|t|^2)^{s/2})^{p'} dt\right)^{1/p'} \left(\int_{\mathbb{R}^N} (1+|t|^2)^{-sp/2} dt\right)^{1/p} \\ &\leq C_{\mathcal{F}}(p) C_0^{1/p} \left(\int_{\mathbb{R}^N} (\|\mathcal{F}^{-1}((1+|\cdot|^2)^{s/2}\hat{g}(\cdot))(t)\|)^p dt\right)^{1/p} \\ &= C_{\mathcal{F}}(p) C_0^{1/p} \|g\|_{H^s_p},\end{aligned}$$

where $C_0 = \int_{\mathbb{R}^N} (1+|t|^2)^{-sp/2} dt$, which is finite, whenever $s > N/p$.

Applying this to $g(t) = m(t)x$ for a fixed $x \in X$, we obtain

$$\int_{\mathbb{R}^N} \|(\mathcal{F}^{-1} m)(s)x\|_Y ds \leq C\|m\|_{H^s_p(\mathbb{R}^N, \mathcal{B}(X,Y))} \|x\|.$$

Since X^* and Y^* also have the same Fourier type p, we obtain in the same manner, using $(\mathcal{F}^{-1}m)(s)^* = (\mathcal{F}^{-1}m(\cdot)^*)(s)$, the estimate

$$\int_{\mathbb{R}^N} ||(\mathcal{F}^{-1}m)(s)^* y^*||_{X^*} ds \le C||m||_{H_p^s(\mathbb{R}^N, \mathcal{B}(Y^*,X^*))}||y^*||.$$

Now we apply Lemma 2.1 to finish the proof. $\square$

Remark 2.4. If $m(s) = n(s)A$, where $n \in H_p^s(\mathbb{R}^N)$ is a scalar-valued function and $A \in \mathcal{B}(X,Y)$, then no Fourier type requirement on X or Y is necessary in Proposition 2.3.

In the following let $\psi_0, \psi_1 \in \mathcal{S}(\mathbb{R}^N)$, such that $\text{supp}\psi_0 \subset \{x \in \mathbb{R}^N : |x| \le 4\}$, $\text{supp}\psi_1 \subset \{x \in \mathbb{R}^N : 1/4 < |x| \le 4\}$, and $\psi_0|_{\{x\in\mathbb{R}^N:|x|\le 2\}} \equiv \psi_1|_{\{x\in\mathbb{R}^N:1/2\le|x|\le 2\}} \equiv 1$

Theorem 2.5. *Let $M : \mathbb{R}^N \to \mathcal{B}(X,Y)$ be such that there exists a constant C with*

$$\begin{aligned} ||\psi_1(\cdot)M(2^k\cdot)||_{H_p^r(\mathbb{R}^N,\mathcal{B}(X,Y))} &\le C \\ ||\psi_0(\cdot)M(\cdot)||_{H_p^r(\mathbb{R}^N,\mathcal{B}(X,Y))} &\le C \end{aligned}$$

for all $k \in \mathbb{N}$. If X and Y both have Fourier type $1 \le p \le 2$ and $r > N/p$ then

$$\widehat{Kf}(t) = M(t)\widehat{f}(t), \quad t \in \mathbb{R}^N,$$

for $f \in \mathcal{S}(\mathbb{R}^N, X)$ can be extended to a bounded operator

$$K : B_{q,r}^s(\mathbb{R}^N, X) \to B_{q,r}^s(\mathbb{R}^N, Y)$$

for all $1 \le q, r \le \infty$, $s \in \mathbb{R}$.

Proof. We define operators K_j for $f \in \mathcal{S}(\mathbb{R}^N, X)$ through

$$(K_j f)(t) = \int_{\mathbb{R}} k_j(t-s)f(s)ds, \quad t \in \mathbb{R}^N$$

where $k_j = \mathcal{F}^{-1}(\psi_1(2^{-j}\cdot)M(\cdot))$ for $j \in \mathbb{N}$ and $k_0 = \mathcal{F}^{-1}(\psi_0 M)$. For all $j \in \mathbb{N}$ we have

$$\mathcal{F}(\mathcal{F}^{-1}(\phi_j^N) * Kf) = \phi_j^N M\widehat{f} = \psi_1(2^{-j}\cdot)M\phi_j^N\widehat{f} = \mathcal{F}(K_j(\mathcal{F}^{-1}(\phi_j^N) * f))$$

and similarly for $j = 0$, so that $||\mathcal{F}^{-1}(\phi_j^N) * Kf||_q \le ||K_j|| \, ||\mathcal{F}^{-1}(\phi_j^N) * f||_q$. Since

$$||Kf||_{B_{q,r}^s}^r = \sum_{j=0}^{\infty}[2^{js}||\mathcal{F}^{-1}(\phi_j^N) * Kf||_q)^r$$

it remains to show that the norms $||K_j||$, $j \in \mathbb{N}_0$ are uniformly bounded. Note that for $j \in \mathbb{N}$ we have $||K_j|| = ||k_j||_1 = ||2^{-j}k_j(2^{-j}\cdot)||_1 = ||k_j||_1$ and

$$2^{-j}k_j(2^{-j}t) = 2^{-j}(\mathcal{F}^{-1}(\psi_1(2^{-j}\cdot)M))(2^{-j}t) = \mathcal{F}^{-1}(\psi_1(\cdot)M(2^j\cdot))(t). \tag{3}$$

Now our assumption on M allows us to apply Proposition 2.3 (and its proof) to assure that $\sup_{j\in\mathbb{N}_0}||K_j|| < \infty$ and the proof is complete. $\square$

Remark 2.6. (a) The first of the conditions on M in Theorem 2.5 may be replaced by

$$||\psi_1(2^{-k}\cdot)M(\cdot)||_{H_p^r(\mathbb{R}^N,\mathcal{B}(X,Y))} \le C$$

as can be seen by the calculation (3) in the proof.

(b) Proposition 2.3, Remark 2.4, and Theorem 2.5 are also true, if we replace $H_p^s(\mathbb{R}^N, \mathcal{B}(X,Y))$ with $B_{p,q}^s(\mathbb{R}^N, \mathcal{B}(X,Y))$ or $W_p^n(\mathbb{R}^N, \mathcal{B}(X,Y))$ with $n > N/p$ by virtue of the inclusions between these spaces.

(c) In the latter case we may omit the auxiliary functions ψ_0 and ψ_1, if we substitute $W_p^n(I_j^N, \mathcal{B}(X,Y))$ for $W_p^n(\mathbb{R}^N, \mathcal{B}(X,Y))$ with $j = 0, 1$, respectively.

Proof. (c) This is straightforward calculation; we will therefore only give the first estimate for the first derivative and $k \geq 3$. Let $D := \{1/4 \leq |x| \leq 4\}$.

$$\begin{aligned}
||(\psi_1 M(2^k\cdot))'||_p &\leq ||M(2^k\cdot)\psi_1'||_p + ||M(2^k\cdot)'\psi_1||_p \\
&\leq \quad ||\psi_1'||_\infty ||M(2^k\cdot)||_{L_p(D)} + ||M(2^k\cdot)'||_{L_p(D)} \\
&\leq \quad ||\psi_1'||_\infty (2^{-2/p}||M(2^{k-2}\cdot)||_{L_p(I_1^N)} + ||M(2^k\cdot)||_{L_p(I_1^N)}) \\
&\qquad + 2^{-2/p}||M(2^{k-2}\cdot)'||_{L_p(I_1^N)} + ||M(2^k\cdot)'||_{L_p(I_1^N)} \\
&\leq \quad 3(||\psi_1'||_\infty + 1)C.
\end{aligned}$$

□

Remark 2.7. The proof of Proposition 2.3 and Theorem 2.5 shows that we may weaken the assumption of the theorem for M by assuming the pointwise conditions

$$\begin{aligned}
||M(2^k\cdot)x||_{W_p^n(I_1^N,Y)} &\leq C||x|| \\
||M(\cdot)x||_{W_p^n(I_0^N,Y)} &\leq C||x|| \\
||M(2^k\cdot)^* y^*||_{W_p^n(I_1^N,X^*)} &\leq C||y^*|| \\
||M(\cdot)^* y^*||_{W_p^n(I_0^N,X^*)} &\leq C||y^*||
\end{aligned}$$

instead of the uniform estimate. (Of course, this pointwise condition may be applied to other spaces, e. g. $H_p^r(\mathbb{R}^N, \mathcal{B}(X,Y))$, as well as to Proposition 2.3 if appropriately modified.)

Corollary 2.8. *Suppose $||(1 + |t|)^{|\alpha|} M^{(\alpha)}(t)||_{\mathcal{B}(X,Y)} \leq C$ for all multi-indices α with $|\alpha| \leq N + 1$ and $t \in \mathbb{R}^N$. (If X and Y are uniformly convex, it is sufficient to consider $|\alpha| \leq N$.) Then*

$$\widehat{Kf} = M\hat{f}$$

for $f \in \mathcal{S}(\mathbb{R}^N, X)$ can be extended to an operator

$$K : B_{p,q}^s(\mathbb{R}^N, X) \to B_{p,q}^s(\mathbb{R}^N, Y)$$

for all $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$.

Proof. Let $M_k = M(2^k\cdot)$ for $k \in \mathbb{N}_0$. We then find

$$|||t|^{|\alpha|} M_k^{(\alpha)}(t)|| = ||2^{k|\alpha|}|t|^{|\alpha|} M^{(\alpha)}(2^k t)|| \leq C.$$

and therefore

$$||M_k^{(j)}||_{L_p(I_1^N,X)} \leq C||| \cdot |^{-j}||_{L_p(I_1^N,X)} \leq C 2^{N+1} \mathrm{vol}(I_1^N).$$

Now apply Remark 2.6. If X is uniformly convex, it has Fourier type $p > 1$ (cf. [3]) and $N/p < N$. □

3. Marcinkiewicz type Multipliers

Before we establish the Theorem of Marcinkiewicz for vector-valued multipliers, we need to establish the notion of bounded variation for functions $f : \mathbb{R}^N \to X$ for a Banach space X. There are many ways of defining such a space. We need however a certain approximation property (Lemma 3.1) of the N-dimensional step-functions. We define

$$BV_N(\mathbb{R}^N, X) := BV(\mathbb{R}, BV(\mathbb{R}, \cdots, BV(\mathbb{R}, X)\cdots))$$

Note that the order of taking variations is important in this definition. This space is handy in the upcoming proofs, although it is more common to define the space of functions of bounded (N-dimensional) variation as follows.

Let $S(N)$ be the symmetric group of order N and

$$\mathcal{Q}_N := \{(a_1, b_1] \times \cdots \times (a_N, b_N] : a_k, b_k \in \mathbb{R}, a_k < b_k; k = 1, \dots, N\}$$

the set of cuboids in $\mathbb{R}^N$. A subset $\mathcal{R} \subset \mathcal{Q}_N$ is called a decomposition of $\mathbb{R}^N$, if for $R, S \in \mathcal{R}$, with $R \neq S$ we have $R \cap S = \emptyset$ and $\cup\mathcal{R} = \mathbb{R}^N$. The set of all decompositions is called $\mathcal{DC}_N$. For any function $f : \mathbb{R}^N \to X$ put

$$\Delta_h^k f = f(x_1, \dots, x_{k-1}, x_k + h, x_{k+1}, \dots, x_N) - f(x),$$

for $k = 1, \dots, N$, and $h \geq 0$ and

$$\Delta_R^\sigma f = \Delta^{\sigma(1)}_{b_{\sigma(1)} - a_{\sigma(1)}} \cdots \Delta^{\sigma(n)}_{b_{\sigma(n)} - a_{\sigma(n)}} f$$

for $R = [a_1, b_1] \times \cdots \times [a_n, b_n] \in \mathcal{Q}_n$ and $\sigma \in S(N)$. For a function $f : R^N \to X$ we define the N-variation as

$$\mathrm{Var}_N(f) := \sup_{\mathcal{R} \subset \mathcal{DC}_N, \sigma \in S(N)} \sum_{R \in \mathcal{R}} \|\Delta_R^\sigma f\| < \infty.$$

Finally, let $f_{\sigma,n} : \mathbb{R}^n \times \mathbb{R}^{N-n} \to X$ be defined by

$$f_{\sigma,n}(x_1, \dots, x_n)(x_{n+1}, \dots, x_N) = f(x_{\sigma(1)}, \dots, x_{\sigma(N)}),$$

if $\sigma \in S(N)$. The space of all functions of bounded variation then is the Banach space $BV(\mathbb{R}^N, X)$ of functions f, for which

$$\|f\|_{BV} = \|f(0)\| + \sum_{\sigma \in S(N), n \leq N} \mathrm{Var}_n(f_{\sigma,n}(\cdot)(0)).$$

is finite.

Lemma 3.1. *Let* $\mathcal{U}_N := \{f = \sum x_i \chi_{R_i} : x_i \in X, R_i \in \mathcal{Q}_N\}$. *We define* $\|f\|_{\mathcal{U}} = \inf\{\sum \|x_i\| : f = \sum x_i \chi_{R_i}\}$ *for* $f \in \mathcal{U}_N$. *Then*

(i) $BV(\mathbb{R}^N, X) \subset BV_N(\mathbb{R}^N, X)$.

(ii) For any $f \in BV_N(\mathbb{R}^N, X)$ *there exists a sequence* $(f_n)_n \subset \mathcal{U}$ *with* $f_n(s) \to f(s)$ *uniformly and* $\|f_n\|_{\mathcal{U}} \leq 3^N \mathrm{Var}_N(f)$.

(iii) If $f \in BV_N(\mathbb{R}^N, X)$ *and* $\phi \in \mathcal{S}(\mathbb{R}^N)$, *then* $\phi f \in BV_N(\mathbb{R}^N, X)$.

Proof. (i): We shall prove the inclusion for $N = 2$ and leave the induction process for $N \geq 3$ to the reader. Let $f \in BV(\mathbb{R}^2, X)$ and choose $\sigma = \mathrm{id}$. Let $g = f_{\mathrm{id}(1)}$.

Then

$$\begin{aligned}
||f||_{BV(\mathbb{R},BV(\mathbb{R},X))} &= ||g(0)||_{BV(\mathbb{R},X)} + \sup_{\mathcal{R}\in\mathcal{Q}_1}\sum_{R\in\mathcal{R}}||\Delta^1_R g||_{BV(\mathbb{R},X)} \\
&\leq |f(0,0)| + \sup_{\mathcal{R}\in\mathcal{Q}_1}\sum_{R\in\mathcal{R}}|\Delta^2_R f(0,\cdot)| \\
&\quad + \sup_{\mathcal{R}_1\in\mathcal{Q}_1}\sum_{R_1\in\mathcal{R}_1}|\Delta^1_{R_1}(|f(\cdot,0)| + \sup_{\mathcal{R}_2\in\mathcal{Q}_1}\sum_{R_2\in\mathcal{R}_2}|\Delta^2_{R_2}f|)| \\
&\leq |f(0,0)| + \sup_{\mathcal{R}\in\mathcal{Q}_1}\sum_{R\in\mathcal{R}}|\Delta^2_R f(0,\cdot)| + \sup_{\mathcal{R}\in\mathcal{Q}_1}\sum_{R\in\mathcal{R}}|\Delta^2_R f(\cdot,0)| \\
&\quad + \sup_{\mathcal{R}\in\mathcal{Q}_2}\sum_{R\in\mathcal{R}}|\Delta^\sigma_R f| \\
&\leq ||f||_{BV(\mathbb{R}^2,X)}.
\end{aligned}$$

For (ii) we again proceed by induction. Assume $N = 1$ and let f be of bounded variation and w.l.o.g. right continuous. Let $\Delta(x) = \lim_{y\searrow x} f(y) - f(x)$. Then for $n \in \mathbb{N}_0$ the set $N_n = \{x : 1/(n+1) \leq |\Delta(x)| < 1/n\}$ is finite, i. e. $N_n = \{x^{(n)}_1, \ldots, x^{(n)}_{k(n)}\}$. Since $f \in BV(\mathbb{R}, X)$, we have that

$$h = \sum_{n=0}^{\infty}\sum_{j=0}^{k(n)} \Delta(x^{(n)}_j)\chi_{(x^{(n)}_j,\infty)}$$

is of bounded variation. Moreover, $g = f - h$ is of bounded variation and continuous. Obviously, there exists $h_n \to h$ with $h_n \in \mathcal{U}_1$, such that $\mathrm{Var}(h_n) \leq \mathrm{Var}(h)$. For $\epsilon > 0$ let $V_n = \sum_{j=0}^{n} |g(x_{j+1}) - g(x_j)|$, such that $|\mathrm{Var}(g) - \widetilde{V}_n| < \epsilon$ for every variation on a partition finer than that on $(x_j)_{j=0}^{n+1}$. Then $g_n = \sum_{j=0}^{n}(g(x_j) - g(x_{j+1})\chi_{(x_j,\infty)}$ has variation V_n, thus $\mathrm{Var}(g_n) < \mathrm{Var}(g) + \epsilon$. Moreover, $g_n \to g$ in the supremum norm, as $\max_{0\leq j\leq n}|x_{j+1} - x_j| \to 0$, $x_0 \to -\infty$ and $x_n \to +\infty$. Thus we obtain a sequence $g_n + h_n \to f$ as $n \to \infty$. It is obvious, that $||(g_n + h_n)||_{\mathcal{U}} \leq 3\mathrm{Var}(f)$ as $n \to \infty$.

For any $f \in BV_N(\mathbb{R}^N, X)$ we find by the induction hypothesis an approximation by step functions $f_n = \sum g_k \chi_{R_k} \in \mathcal{U}_1$ with values g_k in $BV_{N-1}(\mathbb{R}^{N-1}, X)$. Fix $n \in \mathbb{N}$, such that $||f - f_n||_\infty < \epsilon/2$ and let $n_0 \in \mathbb{N}$ be the cardinality of the index range of the sum. Each one of the finite number of step functions g_k may in turn be approximated in the sup-norm by step functions $g_{k,l} = \sum h_m \chi_{S_m} \in \mathcal{U}_{N-1}$ by the induction hypothesis. For each g_k fix a $g_{k,l}$, such that $||g_k - g_{k,l}||_\infty < \epsilon/2n_0$. Since $(R_k)_k$ and $(S_m)_m$ are each subsets of decompositions, we have with $\widetilde{f}_n = \sum\sum g_{k,l}\chi_{R_k}\chi_{S_m}$ that $||f - \widetilde{f}_n||_\infty < \epsilon$. Moreover, $R_k \times S_m \subset \mathcal{Q}_N$, therefore we have an approximating element $\widetilde{f}_n$ in $\mathcal{U}_N$.

Now we verify the bound for the variation. We estimate by induction hypothesis

$$\begin{aligned}
||\widetilde{f}_n||_{\mathcal{U}} &\leq \sum_k\sum_l ||g_{k,l}||_{BV_{N-2}(\mathbb{R}^{N-2},X)} \\
&\leq 3\sum_k ||g_k||_{BV_{N-1}(\mathbb{R}^{N-1},X)} \leq 3^2\mathrm{Var}_2(f),
\end{aligned}$$

which proves the claim.

To prove (iii), a induction in the same spirit yields the claim. For $N = 1$ we have

$$||\phi f||_{BV(\mathbb{R},X)} \leq ||\phi||_\infty ||f||_{BV(\mathbb{R},X)} + \int_{\mathbb{R}} ||\phi'(x)||_\infty dx ||f||_\infty .$$

For $N \geq 2$ a generalization of Leibniz' rule of differentiation is used. □

Lemma 3.2. *If $f : \mathbb{R}^N \to X$ is N-times continuously differentiable and there exists a $C \geq 0$, such that the derivatives satisfy*

$$\int_{\mathbb{R}^n} ||\frac{\partial^n}{\partial x_{\sigma(1)} \cdots \partial x_{\sigma(n)}} f_{\sigma,n} f(\cdot)(0)|| dx_{\sigma(1)} \cdots dx_{\sigma(n)} \leq C$$

for all $\sigma \in S(N)$ and $1 \leq n \leq N$, then $f \in BV(\mathbb{R}^N, X)$.

Proof. Let g be any partial derivative of f up to order $N-1$. Then we obtain $g_{\sigma(i)}(b)(z) - g_{\sigma(i)}(a)(z) = \int_a^b g_{\sigma(i)}{}'(x)(z) dx$. We can use this fact to obtain for any $\sigma \in S(N)$ via induction

$$\Delta_R^\sigma f_{\sigma,n} f = \int_R \frac{\partial^n}{\partial x_{\sigma(1)} \cdots \partial x_{\sigma(n)}} (f_{\sigma,n}(\cdot)(0)) dx_{\sigma(1)} \cdots dx_{\sigma(n)} .$$

It is now easy to calculate the $BV(\mathbb{R}^N, X)$-norm of f. □

We will also need the following elementary facts, combined in a lemma.

Lemma 3.3. *For any Banach space X and any $1 \leq p < \infty$ the following holds:*
(i) The set

$$\{f \in L_1(\mathbb{R}^N, X) \cap L_p(\mathbb{R}^N, X) : \mathrm{supp}\widehat{f} \text{ is bounded } \} \tag{4}$$

is dense in $L_p(\mathbb{R}^N, X)$.
(ii) if $f, \widehat{f} \in L_1(\mathbb{R}^N, X)$ and $g, \widehat{g} \in L_1(\mathbb{R}^N, X^*)$, then

$$\int_{\mathbb{R}^N} < f(t), g(t) > dt = \int_{\mathbb{R}^N} < \widehat{f}(t), \widehat{g}(t) > dt.$$

Proof. (i) This is proved as in the scalar-valued case. (ii) Since $f = \mathcal{F}^{-1}\widehat{f} \in L_\infty(\mathbb{R}^N, X)$ we have with

$$[f, g] := \int_{\mathbb{R}^N} < f(t), g(t) > dt$$

that $[f, g] = [f, \mathcal{F}^{-1}\widehat{g}] = [(\mathcal{F}^{-1})^* f, \widehat{g}] = [\widehat{f}, \widehat{g}]$. □

Proposition 3.4. *Let X or Y have the UMD property and let $M \in BV_N(\mathbb{R}^N, \mathcal{B}(X, Y))$, then*

$$\widehat{Kf}(t) = M(t)\widehat{f}(t), \quad t \in \mathbb{R}^N$$

for $f \in \mathcal{S}(\mathbb{R}^N, X)$ can be extended to a bounded operator

$$K : L_p(\mathbb{R}^N, X) \to L_p(\mathbb{R}^N, Y)$$

for all $1 < p < \infty$.

Proof. Let $M_k(t) = -i\mathrm{sign}(t_k)$. Then the multiplier M_k corresponds to the Hilbert transform $\mathcal{H}_k$ with respect to the kth variable. This operator is bounded, since X has the UMD property. Let $B_k(a,b) = \mathbb{R} \times \cdots \times (a,b] \times \cdots \times \mathbb{R}$, where $a, b \in \mathbb{R} \cup \{\pm\infty\}$, $a < b$ and $(a,b]$ is located in the kth component. Then (a. e.)

$$\begin{aligned} \chi_{B_k(a,b)}(t) &= 1 - \chi_{B_k(-\infty,0)}(t-a) - \chi_{B_k(0,\infty)}(t-b) \\ &= 1 - \frac{1}{2}(1 - iM_k(t-a)) - \frac{1}{2}(1 + iM_k(t-b)) \\ &= \frac{i}{2}(M_k(t-a) - M_k(t-b)). \end{aligned}$$

But we have

$$(\mathcal{F}^{-1}M_k(\cdot - a)\mathcal{F}f)(t) = (\mathcal{F}^{-1}M_k\mathcal{F}(e^{-ia\cdot}f))(t) = \mathcal{H}_k(e^{-ia\cdot}f)$$

Thus $M_{a,b}(t) = \chi_{B_k(a,b)}(t)A$ for a fixed operator $A \in \mathcal{B}(X,Y)$ is a multiplier defining an operator with norm bounded by $\|\mathcal{H}_k\|\,\|A\| = \|\mathcal{H}\|\,\|A\|$.

Now, $\chi_R A := \chi_{B_1(a_1,b_1)} \cdots \chi_{B_N(a_N,b_N)}A$ defines an operator with norm bounded by $\|\mathcal{H}\|^N\|A\|$. If we have

$$M(t) = \sum_{j=0}^{n} \chi_{R_j} T_j, \tag{5}$$

where $(R_j)_{j=0}^n \subset \mathcal{Q}_N$ and $(T_j)_{j=0}^n \subset \mathcal{B}(X,Y)$, then M is a multiplier with the norm of the associated operator bounded by

$$\|\mathcal{H}\|^N \sum_{j=0}^{n} \|T_j\|$$

For general M of bounded variation there exists by Lemma 3.1 a sequence of $(M^{(k)})_k$ of the form (5), which converges uniformly to M and with $\|M^{(k)}\|_{\mathcal{U}} \le 3^N \mathrm{Var}_N(M)$. We have for $f \in L_p(\mathbb{R}^N, X) \cap L_1(\mathbb{R}^N, X)$ and $g \in L_q(\mathbb{R}^N, X^*) \cap L_1(\mathbb{R}^N, X^*)$, $1/p + 1/q = 1$, with $\mathrm{supp}(\widehat{f})$ and $\mathrm{supp}(\widehat{g})$ bounded that

$$\begin{aligned} &|\int_{\mathbb{R}^N} < g(s), \mathcal{F}^{-1}(M(\cdot)\widehat{f})(s) > ds| \\ =\;& |\int_{\mathbb{R}^N} < \widehat{g}(s), M(s)\widehat{f}(s) > ds| \\ =\;& \lim_{k\to\infty} |\int_{\mathbb{R}^N} < \widehat{g}(s), M^{(k)}(s)\widehat{f}(s) > ds| \\ =\;& \lim_{k\to\infty} |\int_{\mathbb{R}^N} < g(s), \mathcal{F}^{-1}(M^{(k)}(s)\widehat{f})(s) > ds| \\ \le\;& 3^N \mathrm{Var}_N(M)\|g\|_q\|f\|_p. \end{aligned}$$

By Lemma 3.3 (i) K_0 extends to a bounded operator, since $L_q(\mathbb{R}^N, X^*)$ is the dual of $L_p(\mathbb{R}^N, X)$ for the reflexive space X. □

Theorem 3.5. *Let X or Y have the UMD property. Assume that the variation of $M : \mathbb{R}^N \to \mathcal{B}(X,Y)$ on the hypercubes $K_{k,\tau}^N$ of the Lizorkin decomposition are uniformly bounded by $C \ge 0$. Then*

$$\widehat{Kf}(t) = M(t)\widehat{f}(t), \quad t \in \mathbb{R}^N,$$

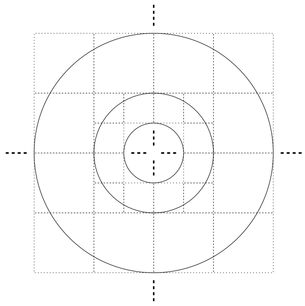

FIGURE 2. In 2 dimensions, I_j^2 is always covered by 24 squares $K_{k,\tau}^2$

for $f \in \mathcal{S}(\mathbb{R}^N, X)$ *can be extended to a bounded operator*

$$K : B_{p,q}^s(\mathbb{R}^N, X) \to B_{p,q}^s(\mathbb{R}^N, Y)$$

for all $1 < p < \infty$, $1 \leq q \leq \infty$, $s \in \mathbb{R}$.

Proof. We first see by geometrical considerations, that for $A(j, N) = \{(\tau, k) : \tau \in \Gamma_N, j - 1 - \log_2(N)/2 \leq k \leq j + 1\}$

$$I_j^N \subset \bigcup \{K_{k,\tau}^N : (k, \tau) \in A(j, N)\}.$$

But $|A(j, N)| \leq 4^N(2 + \log_2(N)/2)$. Thus, for fixed N, each element of the Besov decomposition is enclosed in a fixed number of elements of the Lizorkin decomposition (see Figure 2). By definition of the Besov norms, we need to show that the operators L_j, $j \in \mathbb{N}_0$, given by

$$\begin{aligned} \widehat{L_j f}(t) &= \chi_{I_j^N}(t)\phi_j^N(t)M(t)\hat{f}(t) \\ &= \chi_{\{\bigcup\{K_{k,\tau}^N : (k,\tau) \in A(j,N)\}\}}(t)\phi_j^N(t)M(t)\hat{f}(t) \\ &= \sum_{(k,\tau) \in A(j,N)} \chi_{K_{k,\tau}^N} M(t)\hat{f}(t) \end{aligned}$$

are uniformly bounded in $\mathcal{B}(L_p(\mathbb{R}^N, X), L_p(\mathbb{R}^N, Y))$. Note that we were able to replace I_j^N by the covering of elements of the Lizorkin decomposition, since the support of ϕ_j^N is enclosed in I_j^N.

We proceed to estimate for each $k \in \mathbb{N}_0$ operators in $L_p(\mathbb{R}^N, X)$ of the form

$$\widehat{L'f}(t) = \chi_{K_{k,\tau}^N}(t)\phi_j^N(t)M(t)\hat{f}(t). \tag{6}$$

Now $\chi_{K_{k,\tau}^N} M(t)$ is uniformly bounded by hypothesis and ϕ_j^N is $\mathcal{S}(\mathbb{R}^N)$. By Proposition 3.4 and Lemma 3.1 all operators L' are uniformly bounded. Since only a sum of $|A(j, N)|$ operators of the form L' are needed to express L_j, we have finished the proof. □

Corollary 3.6. *The UMD property is necessary in both Proposition 3.4 and in Theorem 3.5.*

Proof. First choose $N = 1$, $X = Y$, and assume this space does not have the UMD property. In case of Proposition 3.4 choose the multiplier $M(t) = i\text{sign(t)}I$. By definition, M does not have an associated bounded operator.

In the case of Theorem 3.5 let $1 \leq q < \infty$. We choose a norming sequence ϕ_j^1 with with support in

$$\{x \in \mathbb{R} : \frac{5}{4}2^{j-1} < |x| \leq \frac{7}{4}2^j\}$$

It follows that $\phi_j^1|_{J_j} \equiv 1$, with

$$J_j := \{x \in \mathbb{R} : \frac{7}{4}2^{j-1} < |x| \leq \frac{5}{4}2^j\}.$$

Now take a sequence in $(f_n)_n$ in the set (4) with the property $||f_n||_p = 1$, $||\mathcal{H}f_n||_p \geq n$ and $\text{supp}(\widehat{f_n}) \subset (-2^{k_n}, 2^{k_n}]$ for a strictly increasing sequence $(k_n)_n$. Put $g_n(t) = e^{i2^{k_n+3}t} f_n(t)/n$. Then $||g_n||_p = 1/n$ and

$$\text{supp}(\widehat{g_n}) \subset (7 \cdot 2^{k_n}, 9 \cdot 2^{k_n}] \subset J_{k_n+3}.$$

We then define the multiplier $M(t) = i\text{sign}(t - 2^{k_n+3})I$ for $t \in I^1_{k_n+3}$. M has variation 1 on each $I^1_{k_n+3}$, $n \in \mathbb{N}$. Put $g = \sum_{n=1}^\infty g_n$. Taking the supports of $(g_n)_n$ into account we have

$$\begin{aligned} ||g||^q_{B^0_{p,q}(\mathbb{R},X)} &\leq \sum_{n=1}^\infty ||\mathcal{F}^{-1}(\phi^1_{k_n+3}\widehat{g_n})||_p^q \\ &\leq \sum_{n=1}^\infty ||\mathcal{F}^{-1}(\phi^1_{k_n+3}\widehat{g_n})||_p^q \\ &\leq \sum_{n=1}^\infty ||\mathcal{F}^{-1}\phi^1_{k_n+3}||_1 ||g_n||_p^q \\ &\leq \sum_{n=1}^\infty (1/n)^q < \infty. \end{aligned}$$

But if K were the operator associated with M, we would have

$$\begin{aligned} ||Kg||^q_{B^0_{p,q}(\mathbb{R},X)} &= \sum_{n=0}^\infty ||\mathcal{F}^{-1}(\phi^1_{k_n+3}) * Kg_n||_p^q \\ &= \sum_{n=0}^\infty \frac{1}{n^q} ||\mathcal{H}f_n||_p^q \\ &\geq \sum_{n=0}^\infty 1 \\ &= \infty. \end{aligned}$$

Again we have taken into account the supports of ϕ_j^1 and $M\widehat{g_n}$. Clearly K cannot exist. □

We call the decomposition of $\mathbb{R}^N$ by the rectangles

$$D^N_{\epsilon,j} := \{\epsilon x \in \mathbb{R}^N : x_i \in \left\{ \begin{array}{ll} (2^{j_i-1}, 2^{j_i}] & j_i \neq 0 \\ [0,1] & j_i = 0 \end{array} \right\}\}, \quad \epsilon \in \{\pm 1\}^N, \quad j \in \mathbb{N}_0^N$$

the dyadic decomposition. This decomposition is used in the classical Marcinkiewicz Multiplier theorem. We are only able to obtain a vector-valued version if we do not require the associated operator obtained to be an endomorphism.

Corollary 3.7. *Let X or Y have the UMD property. Assume that the variation of $M : \mathbb{R}^N \to \mathcal{B}(X,Y)$ on the rectangles $D^N_{\epsilon,j}$ of the dyadic decomposition are uniformly bounded by $C \geq 0$. Then*

$$\widehat{Kf}(t) = M(t)\widehat{f}(t), \quad t \in \mathbb{R}^N,$$

for $f \in \mathcal{S}(\mathbb{R}^N, X)$ can be extended to a bounded operator

$$K : B^s_{p,q}(\mathbb{R}^N, X) \to B^{s-\delta}_{p,q}(\mathbb{R}^N, Y)$$

for all $1 < p < \infty$, $1 \leq q \leq \infty$, $s \in \mathbb{R}$ and $\delta > 0$.

Proof. The proof follows the lines of the proof of Theorem 3.5. However, the covering of I^N_j now looks like this:

$$I^N_j \subset \bigcup \{D^N_{\epsilon,k} : k \in B(j,N), \epsilon \in \{\pm 1\}^N\},$$

where $B(j,N) = \{k \in \mathbb{N}^N_0 : j - 1 - \log_2(N)/2 \leq \max(k_i) \leq j+1\}$. But the growth of $|B(j,N)|$ can only be estimated to Order $O(j)$, e. g. $|B(j,N)| \leq C_0(j-1) := 2^N(j-1)(2+\log_2(N)/2)$. If we proceed in the proof of Theorem 3.5, let M be the uniform bound for norms of the appropriately modified operators L' in Eq. (6) for the dyadic decomposition. Let $C_1 = \sup_{k\in\mathbb{N}_0} 2^{-k\delta}(k+1)$. We then estimate for $g \in B^s_{p,q}(\mathbb{R}^N, X)$:

$$\begin{aligned}
||Kg||^q_{B^{s-\delta}_{p,q}(\mathbb{R}^N,X)} &\leq \sum_{k=0}^{\infty} 2^{k(s-\delta)} ||\mathcal{F}^{-1}(\phi^N_k) * Kg||^q_p \\
&\leq (C_0 M)^q \sup_k ||\mathcal{F}^{-1}(\phi^N_k)||^q_1 \sum_{k=0}^{\infty} 2^{k(s-\delta)}(k-1)||g||^q_p \\
&\leq C_1 (C_0 M)^q \sup_k ||\mathcal{F}^{-1}(\phi^N_k)||^q_1 \sum_{k=0}^{\infty} 2^{ks}||g||^q_p \\
&\leq \widetilde{C}||g||^q_{B^s_{p,q}(\mathbb{R}^N,X)}.
\end{aligned}$$

This finishes the proof. ∎

4. Applications

We first consider a simple result for analytic semigroups, before we apply the Mikhlin Multiplier theorem to Volterra integral equations. Let $\widetilde{B}^s_{p,q}(I,X) = \{f \in B^s_{p,q}(\mathbb{R}, X) : \text{supp}(f) \subset \overline{I}\}$ for any interval $I \subset \mathbb{R}$. This is a Banach space. An important note: in this section $\widehat{f}$ will denote the Laplace transform $\int_0^\infty e^{-\lambda t} f(t)dt$, not the Fourier transform, of a function f for λ in an appropriate (complex) right half-plane. The following type of maximal regularity was considered by A. Lunardi [5] and H. Amann [1].

Example 4.1. Let A generate the bounded, analytic semigroup $T(\cdot)$. Then the Cauchy problem

$$u'(t) = Au(t) + f(t), \qquad u(0) = u_0$$

with $u_0 \in D(A)$ and $f \in B^s_{p,q}(\mathbb{R}^0_+, X)$ has a solution $u \in B^{s+1}_{p,q}(\mathbb{R}^0_+, X)$ with $Au \in B^s_{p,q}(\mathbb{R}^0_+, X)$.

Proof. Since the required estimates of Theorem 3.5 on $M(t) = A(t-A)^{-1}$ are well-known to be fulfilled for analytic semigroups, the spaces $B^s_{p,q}(\mathbb{R}, X)$ are spaces of maximal regularity for the mapping $f \mapsto u'$. The conditions on the support follows easily from the uniqueness of a solution u. □

Let Y be a Banach space densely embedded in X. We are interested in solving the equation

$$u(t) = (A * u)(t) + f(t), \qquad u(0) = u_0, \tag{7}$$

in a Banach space X, where

$$A \in L_1^{\mathrm{loc}}([0,\infty), \mathcal{B}(Y,X)) := \bigcap_{a>0} L_1([0,a], \mathcal{B}(Y,X))$$

$u_0 \in X$, and $f : \mathbb{R}^0_+ \to X$ with properties to be determined. X, Y, and A in this section will always be assumed to be of this nature. Maximal regularity is achieved, when there is a Banach space E, such that for all $f \in E$ the solution of (7) is also in E. We call S a *solution family* with respect to A, if $S \in C(\mathbb{R}^0_+, \mathcal{B}_s(X)) \cap C(\mathbb{R}^0_+, \mathcal{B}_s(Y))$ and

$$S(t)y = y + (A * S)(t)y, \qquad S(t)y = y + (S * A)(t)y$$

for all $y \in Y$. We call (7) *parabolic*, if $\widehat{A}(\lambda) : Y \to X$ is closed ($\lambda \in \mathbb{C}_+$) and there exists an $M \geq 0$, such that for all $\lambda \in \mathbb{C}_+$ the estimate $||(I - \widehat{A}(\lambda))^{-1}||_{\mathcal{B}(Z)} \leq M$ holds for $Z = X$ and $Z = Y$.

We recall that $A \in L_1^{\mathrm{loc}}(\mathbb{R}^0_+, \mathcal{B}(Y,X))$ is called *k-regular* if there exists a $C > 0$, such that for all $y \in Y$, $n \leq k$ and $\lambda \in \mathbb{C}_+$ holds

$$||\lambda^n \widehat{A}^{(n)}(\lambda)y||_Z \leq C(||y||_Z + ||\widehat{A}(\lambda)y||_Z)$$

for $Z = X$ and $Z = Y$. (We suppress that in the case $Z = Y$ we actually take the part of $\widehat{A}(\lambda)$ in Y.)

We now consider the equation (7), where A is of sub-exponential growth. If the spaces X and Y have a Fourier type $p > 1$, then the following theorem uses weaker assumptions than those that are used in [8], Section 7.

Theorem 4.2. *Let X, Y have Fourier type p. Assume that (7) is parabolic. Let A be 2-regular, if $p = 1$, or 1-regular, if $p > 1$, then (7) has a solution for each $f \in B^s_{p,q}(\mathbb{R}^0_+, Z)$, such that $u \in B^s_{p,q}(\mathbb{R}^0_+, Z)$, where $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, and $Z = X$ or $Z = Y$.*

Moreover, let $a \in L_1^{\mathrm{loc}}(\mathbb{R}^0_+)$ be 2-regular, if $p = 1$, or 1-regular, if $p > 1$. If there exists a $K \geq 0$ with

$$||\widehat{a}(\lambda)(I - \widehat{A}(\lambda))^{-1}||_{\mathcal{B}(X,Y)} \leq K$$

for $\lambda \in \mathbb{C}_+$ and $f = a * g$, $g \in B^s_{p,q}(\mathbb{R}^0_+, X)$, then $u \in B^s_{p,q}(\mathbb{R}^0_+, Y)$.

Proof. Consider the following multiplier

$$M_0(\lambda) := (I - \widehat{A}(\lambda))^{-1}.$$

This will be a multiplier leading to the bounded solution operator S in both $B^s_{p,q}(\mathbb{R}, X)$ and $B^s_{p,q}(\mathbb{R}, Y)$. By assumption, we have that $\|M_0(\lambda)\|_{\mathcal{B}(Y)} \leq M$, independent of $\lambda \in \mathbb{C}_+$. We need to establish bounds for the derivatives of M_0.

$$\begin{aligned}
\|\lambda M_0{}'(\lambda)\| &\leq \|M_0(\lambda)\|\,\|\lambda\widehat{A}'(\lambda)(I-\widehat{A}(\lambda))^{-1}\| \\
&\leq M(\|(I-\widehat{A}(\lambda))^{-1}\| + \|\widehat{A}(\lambda)(I-\widehat{A}(\lambda))^{-1}\|) \\
&\leq M(2M+1). \\
\|\lambda^2 M_0{}''(\lambda)\| &\leq \|M_0(\lambda)\|\,\|\lambda^2\widehat{A}''(\lambda)(I-\widehat{A}(\lambda))^{-1}\| + 2\lambda^2\|\widehat{A}'(\lambda)M_0{}'(\lambda)\| \\
&\leq M(2M+1) + M(2M+1)^2.
\end{aligned}$$

If A is 1-regular, we use the first inequality and if A is 2-regular both.

We now apply Theorem 2.5 to $M_\delta := M_0(\delta + i\cdot)$ for $\delta > 0$. From this we infer the existence of bounded operators K_δ in $B^s_{p,q}(\mathbb{R}, Y)$ with $\|K_\delta\| \leq C_{p,q,s}$ for all $\delta > 0$ with

$$\mathcal{F}(K_\delta f) = M_\delta \mathcal{F}(f).$$

Using the relationship between Fourier and Laplace transform, we find that the operator S is bounded in $B^s_{p,q}(\mathbb{R}, Y)$, where S is the desired solution operator given by

$$(Sf)(t) = e^{\delta t}(K_\delta(e^{-\delta\cdot}f))(t)$$

and letting $\delta \to 0$.

To prove the second part, we see that $\widehat{a}M_0$ is a multiplier for $a{*}S : B^s_{p,q}(\mathbb{R}, X) \to B^s_{p,q}(\mathbb{R}, Y)$ in the same way; replace the estimate $\|(I-\widehat{A})^{-1}\|_{\mathcal{B}(X)} \leq M$ with $\|\widehat{a}(I-\widehat{A})^{-1}\|_{\mathcal{B}(X,Y)} \leq K$.

The result on the supports of the solutions is a trivial consequence of the fact that Eq. (7) is a convolution equation. □

References

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