Remarks on continuous, non-differentiable flows

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Abstract

We shall (try to) show that any continuous flow $\phi : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ is similar to a continuously differentiable flow. That is, there exists a homeomorphism $\pi : \mathbb{C} \to \mathbb{C}$, such that $t \mapsto \pi^{-1}(\phi(\pi(x), t))$ is differentiable for each $x \in \mathbb{C}$ and the derivative is continuous (in $\mathbb{R} \times \mathbb{C}$).

A flow is a continuous mapping $\phi : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$, with $\phi(t, \phi(s, x)) = \phi(t+s, x)$ and $\phi(0, x) = x$ for all $x \in \mathbb{C}$ and $s, t \in \mathbb{R}$. A partial flow is a mapping $\phi : U \to \mathbb{C}$ satisfying the above conditions for some set $[-\epsilon, \epsilon] \times U$ in $\mathbb{R} \times \mathbb{C}$ such that they make sense. A set $\{\phi(t, x) : t \in I\}$ is called a flow line if I is some interval, and if $I = \mathbb{R}$, it is called an orbit. A flow ϕ is called differentiable if $\frac{\partial \phi}{\partial t}(t, x)$ exists, we denote with ϕ' the derivative for brevity.

We denote with S_ϕ the set of stationary points, i. e. $S_\phi = \{x \in \mathbb{C} : \phi(t, x) = x, t \in \mathbb{R}\}$. Then $N_\phi = \mathbb{C} \backslash S_\phi$ is open. A *flow-box* is a set of the form $\phi([-\epsilon, \epsilon], \Gamma)$, where $\Gamma \cap \phi([-\epsilon, \epsilon], \{x\})$ contains at most one point for every $x \in \Gamma$. It is well-known ([Aart,Oversteegen], [Whitney]), that for each $x_0 \in N_\phi$ there exists a flow-box neighborhood $U = \phi([-\epsilon, \epsilon], \Gamma)$ with $x_0 \in \Gamma$ and Γ compact. $\Gamma_U := \Gamma$ is called a cross-section of U. Moreover, i denotes the imaginary unit, $i^2 = -1$ and S_2 denotes the unit sphere in $\mathbb{R}^3$. A curve or an arc is a continuous function into $\mathbb{C}$ with domain $[0, 1]$. An injective curve is called a Jordan curve.

This first lemma shows, that the cross section can be chosen in a nice fashion.

Lemma 0.1. *Each point $x_0 \in N$ has a flow-box neighborhood U with Γ_U being the image of a continuous, injective mapping $\gamma : [0, 1] \to \mathbb{C}$.*

Proof. We first show that Γ_U may be chosen path connected. First we may choose U to be simply connected. Let $x_1, x_2 \in \Gamma_U$. Let $l : [0, 1] \to U$ be some curve connecting x_1 and x_2 inside U. Now $l(t) = \phi(\tau(t), \gamma(t))$ with $\tau(t) \in [-\epsilon, \epsilon]$ and $\gamma(t) \in \Gamma_U$. Thus $\gamma : [0, 1] \to \Gamma_U$ is well-defined. To see that it is continuous, let $t_n \to t_0$ in $[0, 1]$. By compactness of Γ_U there exists a subsequence $\gamma(t_{n_k}) \to \gamma(s_0)$ and $\tau(t_n) \to \tau(s_1)$. Thus

$$l(t_{n_k}) = \phi(\tau(t_{n_k})), \gamma(t_{n_k})) \to \phi(\tau(s_1), \gamma(s_0))$$

as $k \to \infty$. On the other hand, $l(t_{n_k}) \to l(t_0) = \phi(\tau(t_0), \gamma(t_0))$. But by construction of Γ_U this implies $\gamma(s_0) = \gamma(t_0)$ and $\tau(s_1) = \tau(s_0)$.

Assume that the the three points $x_1, x_2, x_3 \in \Gamma_U$ are not mutually connected by a single Jordan curve. In other words, there exists $y \in \Gamma$, $y \notin \{x_1, x_2, x_3\}$, such that each x_j is connected with y by a single arc α_j inside Γ_U and $\alpha_k \cap \alpha_j = \{y\}$ for $k \neq j$. We now take a Neighborhood V of y, such that appropriate arc-restrictions of the three arcs separate V into three disjoint, simply connected sets G_j, where G_j does not border on $\alpha_j \backslash \{y\}$. Now, fix $\phi(1/2, y) \in G_j$, then there exists a neighborhood W of $(1/2, y)$ such that for all $(t, w) \in W$ we have $\phi(t, z) \in \text{int}(G_j)$. Thus, there exists a $z \in \alpha_j \backslash \{y\}$ with $(1/2, z) \in W$. But since G_j does not border on $\alpha_j \backslash \{y\}$, the flow line through z and $\phi(1/2, z)$ must intersect some other α_k, $k \neq j$. This is a contradiction, since $\alpha_1, \alpha_2, \alpha_3$ where subsets of the cross-section.

Thus for a finite number of points in the cross-section Γ_U, there is a Jordan curve passing through these. Using the compactness of the cross-section, we find that the the whole cross section is a Jordan curve. It follows from a simple topological argument, that Γ_U cannot be a closed curve. Thus, Γ_U is the image of some injective $\gamma : [0, 1] \to \mathbb{C}$. □

The next lemma shows that locally a differentiable flow exists, that keeps the boundary point-wise invariant.

Lemma 0.2. *Let $\phi([-\epsilon, \epsilon], \Gamma)$ be a flow-box F with γ the Jordan curve parameterization of Γ. Then there exists a continuously differentiable partial flow $\psi : [-\epsilon, \epsilon] \times F$, such that $\phi(t, x) = \psi(t, x)$ for $(t, x) \in \partial F$.*

Proof. The boundary of F is composed of two parts, one being the two flow lines $\psi([-\epsilon, \epsilon], \{\gamma(0), \gamma(1)\})$ and the other the cross sectional boundary $\phi(\{-\epsilon, \epsilon\}, \Gamma_F)$. We will first construct a continuously differentiable flow that leaves the former set-invariant and the latter point-wise invariant.

Since the boundary of F is a Jordan curve, there is a conformal mapping $\Omega : F \to Q$, (continuously differentiable in the interior and continuous on the boundary), where $Q = [-a, a] \times [0, 1]$ is a rectangle, with the following properties: $\Omega(-\epsilon, \gamma(0)) = (-a, 0)$, $\Omega(\epsilon, \gamma(0)) = (a, 0)$, $\Omega(-\epsilon, \gamma(1)) = (-a, 1)$, and $\Omega(\epsilon, \gamma(1)) = (a, 1)$ [Pommerenke]. Since we are not interested in the conformality of the mapping Ω, only in its (continuous) differentiability, we may assume $a = \epsilon$ by adding a rescaling mapping.

We define the partial flow ψ through its flow lines. In Q we take the straight lines that connect $\Omega(\phi(-\epsilon, \gamma(s)))$ and $\Omega(\phi(\epsilon, \gamma(s)))$ (see Fig. 1). Since γ is a bijection on Γ, these straight lines represent a partial flow on Q. The flow lines of ψ are now the images under Ω^{-1}. Thus ψ is a well-defined, partial flow, and continuously differentiable with respect to the first variable, since Ω is continuously differentiable. It is also easy to check, that the (reduced) boundary conditions are satisfied.

We can now reparameterize ψ, such that the flow line boundary is not only set-invariant but point-wise invariant. The (unique) reparameterization on the boundary flow lines themselves consists of two continuous, bijective $[-\epsilon, \epsilon]$-valued functions on $[-\epsilon, \epsilon]$. They can be approximated by a continuum of dif-

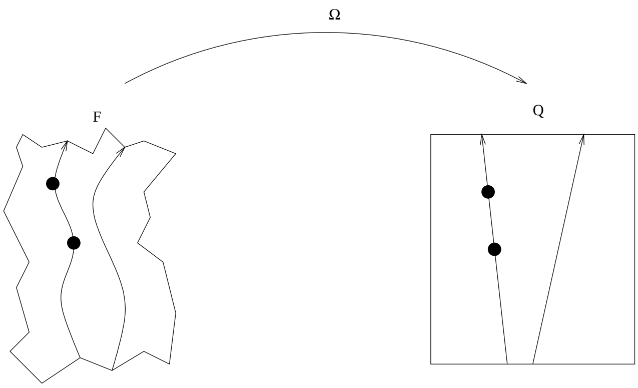

Figure 1:

ferentiable, bijective functions with the same range and domain. These may be used as reparameterization of the flow in the interior of F. □

The following lemma reduces each local flow to the trivial flow. It may convenient to have such a reduction, although presently I can only see use in it in future streamlining of proofs.

Lemma 0.3. *Let* $\phi : [-1,1] \times Q$ *be a partial flow in* $Q = [-1,1] \times [0,1]$*, such that* $\phi(\{\delta\}, [0,1]) = \{\delta\} \times [0,1]$ *for* $\delta = \pm 1$ *and* $\phi([-1,1], \{\delta\}) = [-1,1] \times \{\delta\}$ *for* $\delta = 0, 1$*. Then* ϕ *is similar to the trivial partial flow* τ *on* Q*, i. e.* $\tau(t,x) = x + t(1,0)$.

Proof. Let Γ be the cross section for ϕ given as $\{-1\} \times [0,1]$. Define $\Phi : Q \to Q$ as $\Phi(x) = \phi(t,g)$ for $x = (t,g) \in Q$. It is continuous and bijective. It is therefore a homeomorphism and maps τ into ϕ. □

This lemma is needed to smooth the cross sectional boundaries, after everything else will have been smoothed.

Lemma 0.4. *Given* $u_j \in C([0,1], \mathbb{C})$ *with* $\Re(u_j)(x), \Re(u_j)(x) > 0$ *for* $x \in [0,1]$ *and* $u_j(0) = u_j(1) = 1$ *for* $j = 0, 1$*, there exists a continuously differentiable partial flow* $\psi : [0,1] \times [0,1]^2 \to [0,1]^2$ *with: (i)* $\psi(0,\cdot)$ *leaves the boundary of* $[0,1]^2$ *point-wise invariant and (ii)* $\psi'(j,\cdot) = u_j$.

Proof. Let ψ_0 be the trivial flow along the flow lines $\{(t,x) : t \in [0,1]\}$ $(x \in [0,1])$. We will establish a continuously differentiable partial flow, that matches the boundary condition, fulfills ${\psi_1}'(j,1/2) = u_j(1/2)$, and

$$||{\psi_{n+1}}'(t,x) - {\psi_n}'(t,x)||_\infty \leq \max_{j=0,1} |{\psi_{n+1}}'(j,1/2) - {\psi_n}'(j,1/2)| \qquad (1)$$

for $(t,x) \in [0,1]^3$ and $n = 0$. This can be achieved by taking any function $f : [0,1] \to \mathbb{R}$, that: (i) is strictly bounded by below by $u(t) \equiv 0$ and above by $o(t) \equiv 1$, (ii) with $f'(j) = \Im(u_j(1/2))/\Re(u_j(1/2))$ for $j = 0, 1$, and (iii) $\|f' - 1\|_\infty \leq |f'(j) - 1|$. By taking such f as a flow line between the points $(0, 1/2)$ and $(1, 1/2)$, taking the straight lines between the points $(0, j)$ and $(1, j)$, $(j = 0, 1)$ and taking simple convex combinations in the two separated parts of the square we obtain ψ_1. (The flow keeps the lines $(t, [0, 1])$, $t \in [0, 1]$ invariant as time varies.)

We reiterate this scheme in both of the two separated regions. u and o will be given by either the boundary line of the square or the flow line generated by f. Indeed, this process may be repeated to obtain a sequence of partial flows. A partial flow ψ_{n+1} in that sequence satisfies: (i) the flow lines at $\{(0, k2^{-n-1}) : k = 1, 3, \ldots, 2^{n+1} - 1\}$ are strictly separated by the flow lines at $K_n := \{(0, k2^{-n}) : k = 0, 1, \ldots, 2^n\}$, which coincide with those of ψ_n, (ii) $\psi_{n+1}{}'(j, x) = (1, \Im(u_j(x))/\Re(u_j(x)))$ for $x \in K_n \cup ((1, 0) + K_n)$, and (iii) equation (1) is satisfied. The derivatives of ψ_n converge uniformly to some continuous function $\widetilde{\psi}'$, since u_j is continuous $(j = 0, 1)$. In fact, also ψ_n converges uniformly to $\widetilde{\psi}$. Thus ψ is continuously differentiable with respect to t (with derivative $\widetilde{\psi}'$).

We now have to have to reparameterize $\widetilde{\psi}$ by some continuous function ϕ, continuously differentiable in t, such that $\frac{\partial \phi(t,x)}{\partial t}|_{t=j} = \Re(u_j(x))$ for $j = 0, 1$. We then obtain the desired partial flow ψ. □

Theorem 0.5. *Every continuous flow in $\mathbb{C}$ is similar to a differentiable flow in $\mathbb{C}$.*

Proof. Since N_ϕ is locally compact it can be covered by a countable number of the flow-box neighborhoods $(F_x)_{x \in N_\phi}$, which we call $(F_n)_n$. We proceed by induction. A flow ψ_1 that is differentiable in $\mathrm{int}(F_1)$ that is homeomorphic to ϕ in F_1 via π_1 can be constructed as seen by the previous lemmas. We "surgically" replace ϕ with ψ on F_1. The boundary conditions guarantee that π_1 on F_1 can be extended (trivially) to a homeomorphism on $\mathbb{C}$.

Suppose we have constructed a flow ψ_n which is differentiable in $\widetilde{F}_n := \cup_{k \leq n} F_k$ that is homeomorphic to ϕ via π_n with the additional property: ψ_n has the same values on the (finite number of) cross sections on the boundary of $\widetilde{F}_n$ and coincides with ϕ outside of $\widetilde{F}_n$. We then replace F_{n+1} by a (finite) collection of flow-boxes $(G_j)_j$, which have the properties: each G_j has at most two non-connected cross sectional boundaries in common with $\widetilde{F}_n$, the members of the collection $\mathcal{C} = (G_j)_j \cup \widetilde{F}_n$ have pairwise disjoint interior and $\cup G_i = F_{n+1} \backslash \widetilde{F}_n$ (see Fig. 2). On each of the G_i we replace ϕ with a partially differentiable flow. We thus obtain by the same method as before a flow, that is differentiable on the union of the interiors of $\mathcal{C}$ and otherwise coincides with ϕ.

In the next step we make the flow differentiable on the remaining, finite number of flow lines inside of $\widetilde{F}_{n+1}$. To this end, we take a small (in comparison with the distances between the remaining non-differentiable flow lines) cross section at some point at each flow line and repeat the procedure. This time, the boundary already contains only differentiable flow lines, wherefore the only non-differentiable points remaining are on the cross-sectional boundaries.

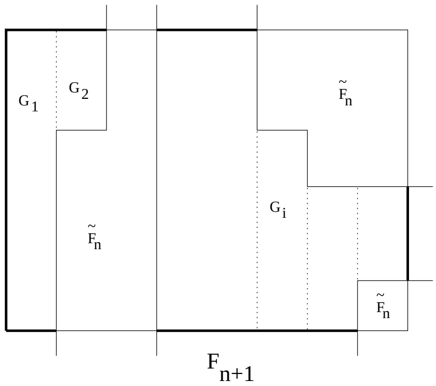

Figure 2:

We take a small (in comparison with the distances between the remaining cross sectional boundaries) flow-box around each cross section and map it homeomorphically onto $Q = [-1,1] \times [0,1]$. We now use Lemma 0.4 and take the inverse image as a local replacement for the previous flow.

In conclusion we have completed the induction. We obtained a sequence of flows ψ_n which converge uniformly on every compact set in N_ϕ by construction. Moreover, we may require the original flow-boxes at each $x \in N_\phi$ to have a diameter of at most $\text{dist}(x, S_\phi)$ without loss of generality. Then we find that the corresponding homeomorphisms π_n on $\mathbb{C}$ do not only remain fixed on each compact set in N_ϕ for $n \geq n_0$, but they also converge uniformly to some π for each compact set in $\mathbb{C}$, where π will be the identity on S_ϕ. We may compactify $\mathbb{C}$ by adding $\{\infty\}$ and treat this point just as a fixed point before. We thus have proved the theorem. □

Corollary 0.6. *Every continuous flow in S_2 is similar to a differentiable flow in S_2.*

Corollary 0.7. *For every $\epsilon > 0$ and every continuous flow ϕ in $\mathbb{C}$ (or S_2) there exists a C^∞ flow ψ in $\mathbb{C}$ (or S_2) that is similar to ϕ, such that (i) $||\phi - \psi||_\infty < \epsilon$ and (ii) if $\psi(t,\cdot) = \pi^{-1}\phi(t, \pi(\cdot))$ then $||\pi - id||_\infty < \epsilon$.*

Proof. In all proofs the differentiability relies on the local mappings, which stem either from a change of parameter, conformal change of domain or some continuous shifting inside a square. All these operations can be made to be C^∞. The patching of these local mappings to a global one is included in that scheme.

To see that the differentiable flow ψ can be chosen arbitrarily close to ϕ, all that is needed is to choose the local flow boxes with a diameter less than $\epsilon/2$. This will yield both (i) and (ii). □

Of course, the theorem and the corollaries remain true, if we replace $\mathbb{C}$ or S_2 with any subset M thereof. Differentiability however can only be obtained in

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the interior of M.